Reconstruct a constant polynomial average

evolve the Riemann Problem

calculate averaged

goto ①
where are we at?

- linear, nonlinear scalar problems
  - use F.D.U. or Godunov or LLF
  - "first order"

- for higher order:
  - linear PDE use REA or linear reconstruction
  - nonlinear PDE Godunov, but approximate the Riemann problem

- systems later
let \( u_t + a u_x = 0 \) + periodic

Linear reconstruction

1. \( u_n = \text{average} \)
   \[
   f_n(x) = \text{linear} \quad = u_n + \sum_n \delta_k (x - x_n) \]
   some slope

2. Evolve
   \[
   u(k_{l+1}) = u(x-a t, t e)\]
\[ u_{k+1} = \frac{1}{h_x} \int_{x_k-Y_k}^{X_k-Y_k} u(x, t+1) \, dx \]

\[ = \frac{1}{h_x} \int_{x_k-Y_k}^{X_k-Y_k} u_k(x-x_k) + \delta_{k-1} (x-x_{k-1}) \, dx \] \quad (1)

\[ + \frac{1}{h_x} \int_{x_k-Y_k}^{X_k-Y_k} u(x-Y_k, t) + \delta_{k} (x-x_k) \] \quad (2)
Algorithm 6.1: non-limited linear reconstruction scheme for linear advection

**Input:** $u_{\ell}$, grid function at time $t_{\ell}$
- $\omega$, weight parameter in $[-1, 1]$
- $a$, advection speed
- $h_t, h_x$, grid spacing
- $\Omega$, list of finite-volume cells

**Output:** $u_{\ell+1}$, grid function at time $t_{\ell+1}$

1. **for each** cell $\Omega_k$
2. \[ \delta_{k,\ell} = \frac{1}{2}(1 + \omega) \Delta^- u_{k,\ell} + \frac{1}{2}(1 - \omega) \Delta^+ u_{k,\ell} \] \{compute slope\}
3. \[ u_{k,\ell+\frac{1}{2}} = u_{k,\ell} - \frac{h_t}{2} a \frac{\delta_{k,\ell}}{h_x} \] \{compute cell value at intermediate time $t_{\ell+\frac{1}{2}}$\}
4. \[ u^-_{k+\frac{1}{2}} = u_{k,\ell+\frac{1}{2}} + \frac{\delta_{k,\ell}}{2} \]
   \[ u^+_{k-\frac{1}{2}} = u_{k,\ell+\frac{1}{2}} - \frac{\delta_{k,\ell}}{2} \] \{compute linear reconstructions at cell interfaces\}
5. \[ u_{k,\ell+1} = u_{k,\ell} - \frac{h_t}{h_x} \left( a u^-_{k+\frac{1}{2}} - a u^+_{k-\frac{1}{2}} \right) \] \{use upwind flux $f^* = au$\}
\[ S = 0 \quad \text{constant or \"Godunov\"} \]

\[ S = \frac{u_{k+1} - u_n}{h_x} \quad \text{Lax-Wendroff} \]

\[ S = \frac{u_k - u_{k-1}}{h_x} \quad \text{Beam-Warm}

\[ S = \frac{u_{k+1} - u_{k-1}}{2h_x} \quad \text{Fromm} \]
need: limit slopes

look at ratios of slopes:

\[ r_k = \frac{u_k - u_{k-1}}{u_{k+1} - u_k} \]

Seek \( \phi(\cdot) \) so that

\[ u_{k+1}^- = u_k + \frac{1}{2} \phi(r_k) (u_{k+1} - u_k) \]

\[ u_{k+1}^+ = u_{k+1} + \frac{1}{2} \phi(r_k) (u_{k+2} - u_{k+1}) \]

\[ f^+(u_{k+1}^-, u_{k+1}^+) \]
New measure: total variation

Consider \( u(x) \)

Total variation in \( u = TV (u(x), \nabla) \)

\[
TV = \int \left| \frac{\partial u}{\partial x} \right| \, dx
\]

We say Total Variation Diminishing if

\[
\frac{d}{dt} \int_{x(t)} (x_2(t) - x_1(t)) \left| \frac{\partial u(\mathbf{x}, t)}{\partial x} \right| \, dx \leq 0
\]

\( x_1(t) \) and \( x_2(t) \) are chosen so as \( \frac{d}{dt} \) above is finite of

\( u_t + (f(u))_x = 0 \)
Smooth function $u(x)$

Then the TV is given by

the extrema: $TV(u) = |u(x_b) - u(x_a)| + |u(x_c) - u(x_b)| + |u(x_d) - u(x_c)| + |u(x_e) - u(x_d)|$
no new extrema between \( x_1, x_2 \)

\[ dTV \left( u(x, t_1), n(t_1) \right) = TV\left( u(x, t_1), n(t_2) \right) \]

\[ \frac{dT V}{dt} = 0 \]
at $t_1 = a + t_2$

$\frac{dT_U}{dt} < 0$
Discrete analog:

Let \( u_{e} = \begin{bmatrix} u_{1, e} \\ u_{2, e} \\ \vdots \\ u_{N, e} \end{bmatrix} \) and periodic

\[ TV(u_{e}) = \sum_{k=1}^{N} \left| u_{k, e} - u_{k-1, e} \right| \]

\( \Rightarrow TV(u_{e}) \) is TVD if

\[ TV(u_{e+1}) \leq TV(u_{e}) \]
Definition 6.15: Linear Scheme

A numerical scheme

\[ u_{k,\ell+1} = \sum_{j} c_j u_{k-j,\ell} \tag{6.116} \]

applied to the conservation law in Equation (6.2) with linear flux function \( f(u) = au \) is called a linear scheme if all coefficients \( c_j \) are constant — i.e., they do not depend on the approximation \( u_{\ell} \) at time \( t_{\ell} \). Otherwise, the scheme is called nonlinear.

Godunov’s theorem (which we state without proof) establishes that linear schemes of order higher than one cannot be TVD:

Theorem 6.16: Godunov’s Theorem

Linear TVD schemes are at most first-order accurate.
$$r_k = \frac{u_k - u_{k-1}}{u_{k+1} - u_k}$$

What do we need for $\phi(r)$?

$$u_{k+1/2} = u_k + \frac{\phi(r_k)}{2} (u_{k+1} - u_k)$$

If $\text{sign}(u_k - u_{k-1}) = \text{sign}(u_{k+1} - u_k)$

then $\phi(r_k) \cdot (u_{k+1} - u_k)$

$$\phi(r_k) = \min (u_k - u_{k-1}, u_{k+1} - u_k)$$

if $\text{sign}(u_k - u_{k-1}) \neq \text{sign}(u_{k+1} - u_k)$

then want $\phi(u_{k+1} - u_k) = 0$

$$\phi(c) = \max (0, \min (c, 1))$$
What conditions on $\phi(r)$ give TVD?
\[ \phi(r) = 2r \]

\[ \phi(r) = 2 \]

\[ \phi = (r + 1)/2 \] 

Fromm

\[ \phi = 1, \text{ Lax-Wendroff} \]

TVD + 2\textsuperscript{nd}-order