$$-u_{xx} = f \text{ on } [0,1]$$
$$u(0) = 0$$
$$u'(1) = 0$$

Consider some space V — space of functions

$$\left[\int_0^1 (f + u_{xx}) v \, dx = 0 \right.$$

Find $u \in V$ s.t.

$$\int_0^1 -u_{xx} v \, dx = \int_0^1 f v \, dx$$

I.B.P. $\Rightarrow \int_0^1 u_x v_x \, dx - u_x v \Big|_0^1 = \int_0^1 f v \, dx$

impose $v(0) = 0 \ \forall v \in V$

& $u_x = 0$

$$\Rightarrow \underbrace{\int_0^1 u_x v_x \, dx}_{a(u,v)} = \int_0^1 f v \, dx$$
$$= \langle f, v \rangle$$

inner product

$$V = \{ v \in L^2([0,1]) \mid a(v,v) < 0, \ v(0) = 0 \}$$

$$a(u,v) = \int_0^1 u_x v_x \, dx$$

bilinear form:

- $a(u,v) = a(v,u)$
- $a(\alpha u + \beta w, v) = \alpha a(u,v) + \beta a(w,v)$
- $a(u,u) \geq 0$ and $a(u,u) = 0$ iff $u = 0$

$$a(u,u) = \int_0^1 (u')^2\,dx$$

$$u(x) = \int_0^x u'(s)\,ds$$

$$\Rightarrow \quad |u(x)| \le \int_0^x |u'(s)|\,ds$$

$$\le \int_0^1 |u'(s)|\,ds$$

$$\Rightarrow \int_0^1 |u(x)|\,dx \le \int_0^1 |u'(s)|\,ds$$

$$\le \left(\int_0^1 1^2\right)^{1/2} \left(\int_0^1 (u'(x))^2\,dx\right)^{1/2}$$

Hölder

$$\le \left(\int_0^1 (u'(x))^2\right)^{1/2}$$

$a(u,v)$ is an inner product.

find $u \in V$ s.t.

$$a(u,v) = \langle f, v \rangle \quad \forall\, v \in V$$

$$V = \{ v \in L^2 \mid a(u,v) < \infty,\ v(0) = 0 \}$$

look at finite dimensional $V^h \subset V$

$$\cancel{V = \{ C^2([0,1]),\ v(0) = 0 \}}$$

<u>Example</u>

$$f = \frac{\pi^2}{4} \sin\left(\frac{\pi x}{2}\right)$$

$$u^*(x) = \sin\left(\frac{\pi x}{2}\right)$$

for $\begin{cases} -u_{xx} = f \\ u(0) = u'(1) = 0 \end{cases}$

where should we look for solutions?

Definition 4.3: Ritz-Galerkin approximation

Let $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ be a bilinear form, and let $\mathcal{V}^h$ be a finite-dimensional subspace of $\mathcal{V}$. Consider the weak form restricted to $\mathcal{V}^h$: Find $u^h \in \mathcal{V}^h$ such that

$$a(u^h, v) = \langle f, v \rangle \ \forall v \in \mathcal{V}^h. \tag{4.12}$$

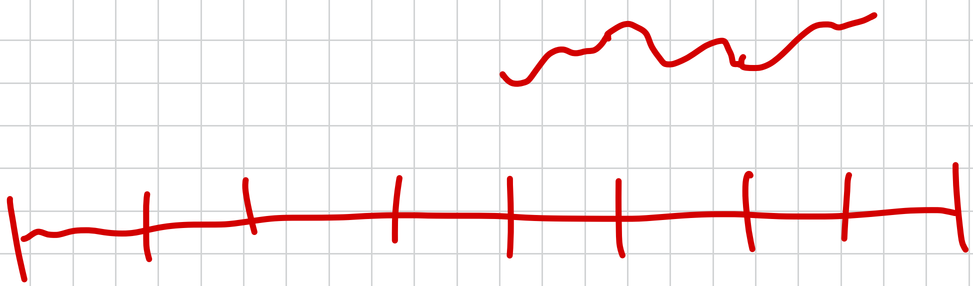
Here, u^h is called the Ritz-Galerkin approximation of the weak solution $u \in \mathcal{V}$.

Example

$$u^h = a_0 + a_1 x + a_2 x^2$$

$$u^h \in V^h \subset V$$

$$= \left\{ v \in L^2 \;\middle|\; \begin{array}{l} a(u,v) < \infty \\ u(0) = 0 \end{array} \right\}$$



$$u^h = a_0 + a_1 x + a_2 x^2$$

$$v(0) = 0 \Rightarrow a_0 = 0$$

$$a(u^h, v) = \langle f, v \rangle \quad \forall\, v \in V^h$$

$$V = \{x, x^2\}$$

$$\int_0^1 (\underbrace{u^h}_{a_1 x + a_2 x^2})_x \cdot 1 = \int_0^1 \frac{\pi^2}{4} \sin\left(\frac{\pi x}{2}\right) \cdot x \, dx$$

and

$$\int_0^1 (\underbrace{u^h}_{a_1 x + a_2 x^2})_x \cdot 2x = \int_0^1 \frac{\pi^2}{4} \sin\left(\frac{\pi x}{2}\right) x^2 \, dx$$

find $u^h \in V^h$ s.t.

$$a(u^h, v) = \langle f, v \rangle \quad \forall\, v \in V^h$$

① does u^h exist?

② is it unique?

③ is it accurate?

Theorem 4.5: Existence and uniqueness

If $f \in L^2([0,1])$, then there exists a unique solution, u^h, to the Ritz-Galerkin approximation in Definition 4.3 when $\mathcal{V}^h \subset \mathcal{V}$ is finite-dimensional.

if $f \in L^2$, $V = \{ v \in L^2 \mid a(u,v) < \infty, \; v(0) = 0 \}$

then $\exists\, u^h$ s.t.

$$a(u^h, v) = \langle f, v \rangle \quad \forall v \in V^h.$$

$$\int_0^1 (f - (u^h)')v' \, dx = 0$$

$V^h \subset V$

let $\phi_1 \ldots \phi_n$ span V^h

$\rightarrow u^h = \sum \alpha_i \phi_i$

find $u^h \in V^h$ s.t.

$a(u^h, v) = \langle f, v \rangle \quad \forall \cancel{v \in V^h}$

$\forall \ v = \phi_i$

Ritz-Galerkin: find $\underline{\alpha}$ s.t.

$$a\left(\sum \alpha_i \phi_i , v \right) = \langle f, v \rangle \quad \forall \ v = \phi_i$$

$$\rightarrow a\left(\sum \alpha_i \phi_i , \phi_j \right) = \langle f, \phi_j \rangle \quad \forall \ \phi_j$$

→ find $\underline{\alpha}$ s.t.

$$\sum \alpha_i \, a(\phi_i, \phi_j) = \langle f, \phi_j \rangle \quad \forall \phi_j$$

$$\begin{bmatrix} a(\phi_1,\phi_1) & a(\phi_1,\phi_2) & \cdots & a(\phi_1,\phi_n) \\ & \ddots & & \\ & & \ddots & \\ & & & a(\phi_n,\phi_n) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \langle f,\phi_1\rangle \\ \langle f,\phi_2\rangle \\ \vdots \\ \langle f,\phi_n\rangle \end{bmatrix}$$

Lemma 4.6: Orthogonality relation

If $u \in \mathcal{V}$ is the solution of the weak form in Equation (4.7) and $u^h \in \mathcal{V}^h$ is the solution of the Ritz-Galerkin approximation from Definition 4.3, then $a(u - u^h, v) = 0, \forall v \in \mathcal{V}^h$.

u^h be the Ritz-Galerkin approx:

$$a(u^h, v) = \langle f, v \rangle \quad \forall v \in V^h.$$

u be the weak solution:

$$a(u, v) = \langle f, v \rangle \quad \forall v \in V$$

$V^h \subset V$

$$\rightarrow a(u, v) = \langle f, v \rangle \quad \forall v \in V^h$$

$$\Rightarrow a(u - u^h, v) = 0 \quad \forall v \in V^h$$

Lemma 4.7: Céa's Lemma

If $u \in \mathcal{V}$ is the solution of the weak form and $u^h \in \mathcal{V}^h$ is the solution of the Ritz-Galerkin approximation, then $\|u - u^h\|_e = \min_{v \in \mathcal{V}^h} \|u - v\|_e$.

$\|u\|_e = \|u\|_{a(\cdot,\cdot)} = \sqrt{a(u,u)}$

Definition 4.8: Approximation assumption

Given $\mathcal{V}^h \subset \mathcal{V}$, the approximation assumption for $\mathcal{V}^h$ is that $\exists\, \epsilon > 0$ such that $\forall w \in$ $C^2([0,1]) \cap \mathcal{V}$, $\min_{v \in \mathcal{V}^h} \|w - v\|_e \leq \epsilon \|w''\|$.

$w \in C^2 \cap V$

$\min \|w - v\|_e \leq \varepsilon \|w''\|$

$O(h^p)$