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$$V = \{ v \in C^2 \mid a(u,v) < \infty, \; v(0) = 0 \}$$

strong form: find $u \in C^2([0,1])$

$$-u_{xx} = f$$

$$u(0) = u'(1) = 0$$

weak form: find $u \in V$ s.t.

$$\int_0^1 u_x v_x \, dx = \int_0^1 f v \, dx \quad \forall v \in V$$

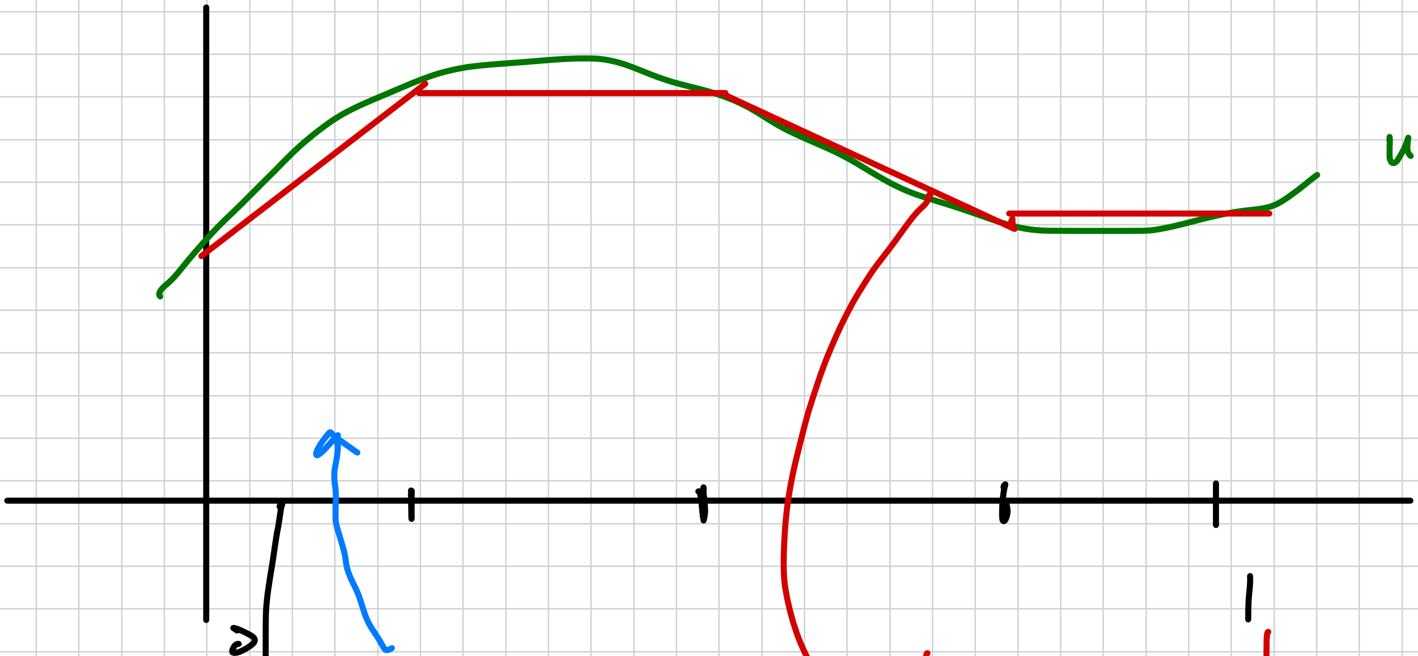
let $V^h \subset V$

finite dimensional

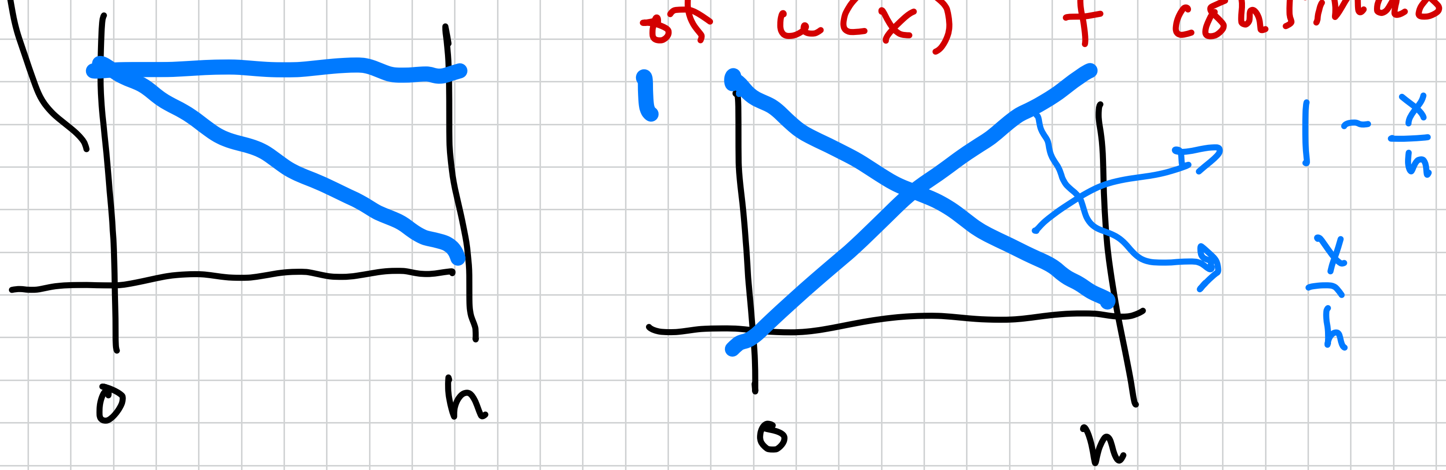
discrete weak form: find $u^h \in V^h$ s.t.

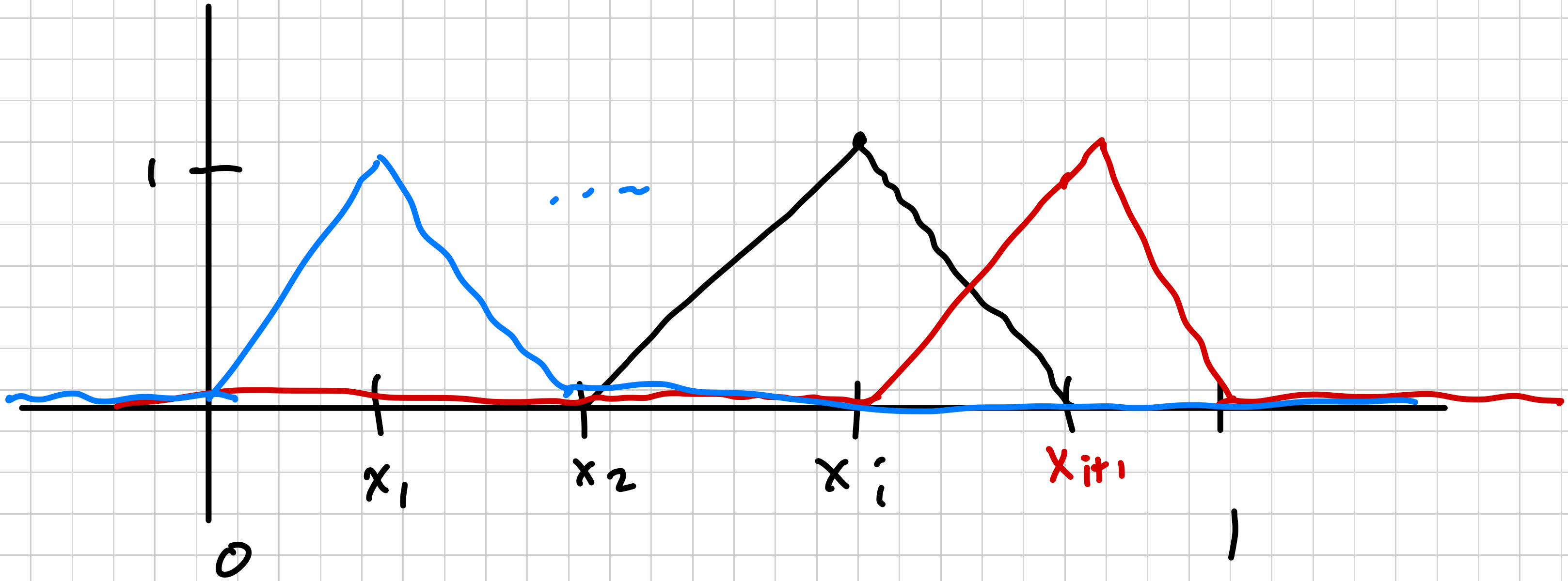
$$\int_0^1 u^h_x v_x \, dx = \int_0^1 f v \, dx \quad \forall v \in V^h$$

What should we pick for V^h?



u^h = piecewise linear interpolant of $u(x)$ + continuous

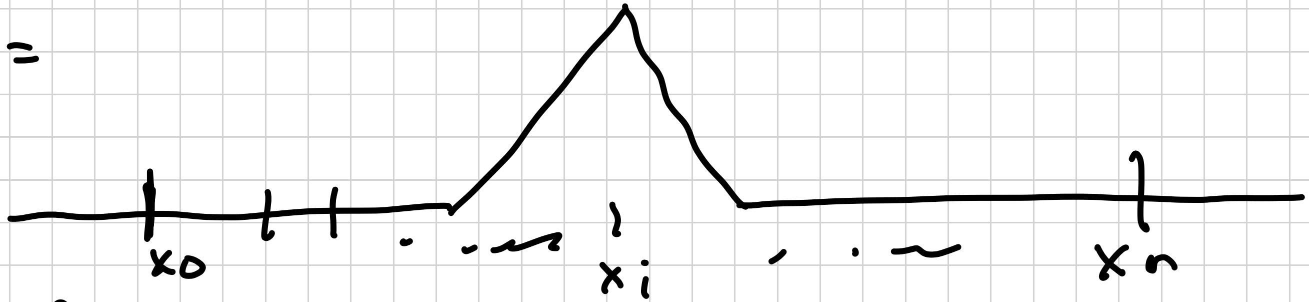




$\phi_i(x)$ = pw linear, continuous

$$\phi_i(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$$

let $\phi_i(x) =$



let $u^h = \sum_{i=0}^{n} u_i \, \phi_i(x)$

find u^h s.t.

$$a(u^h, \phi_j) = \langle f, \phi_j \rangle \quad \forall \phi_j$$

$$\int_0^1 u^{h\prime} \phi_j' \, dx = \int_0^1 f \, \phi_j \, dx$$

$$\int_0^1 \sum_{i=0}^{n} u_i \phi_i' \, \phi_j' \, dx = \int_0^1 f \, \phi_j \, dx \quad \forall \phi_j$$

$$0 \qquad \int_0^1 \sum_{i=0}^{n} u_i \phi_i' \phi_0' \, dx = \int_0^1 f \phi_0 \, dx$$

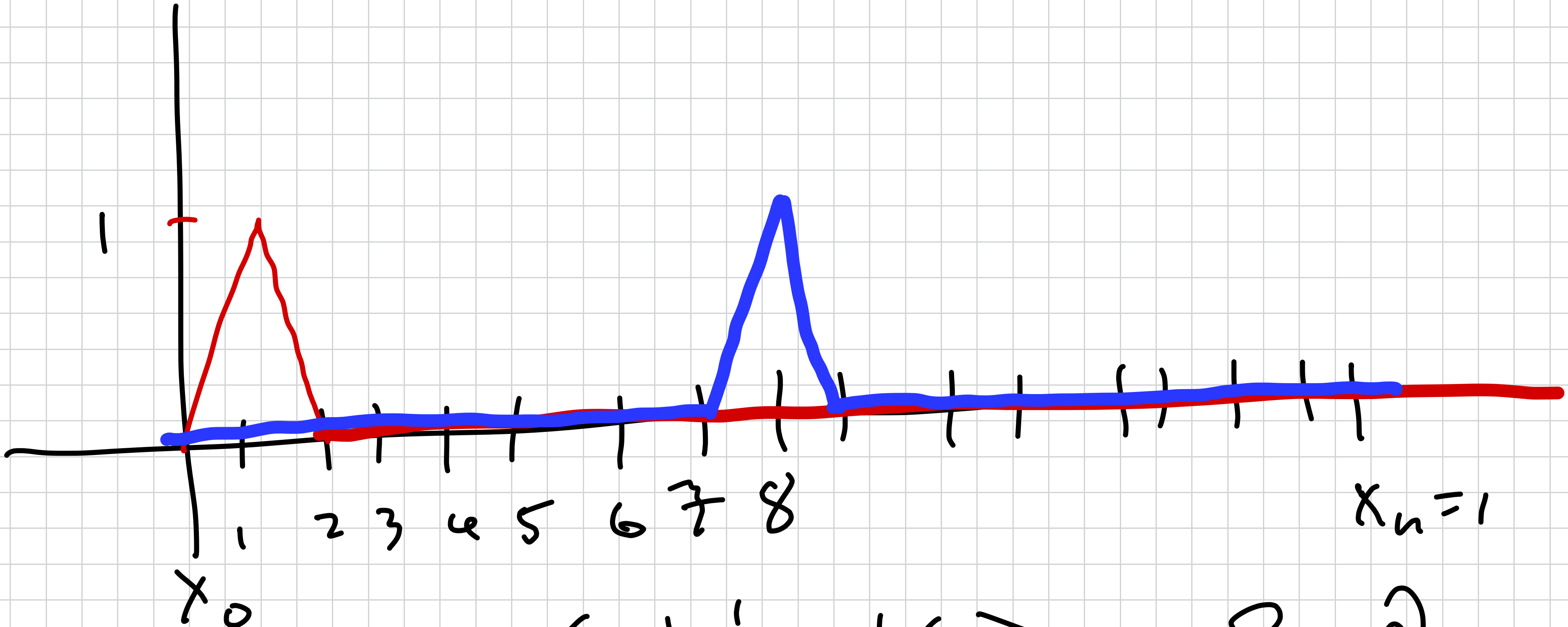
$$1 \qquad \int_0^1 \sum_{i=0}^{n} u_i \phi_i' \phi_1' \, dx = \int_0^1 f \phi_1 \, dx$$

$$\vdots \qquad \vdots$$

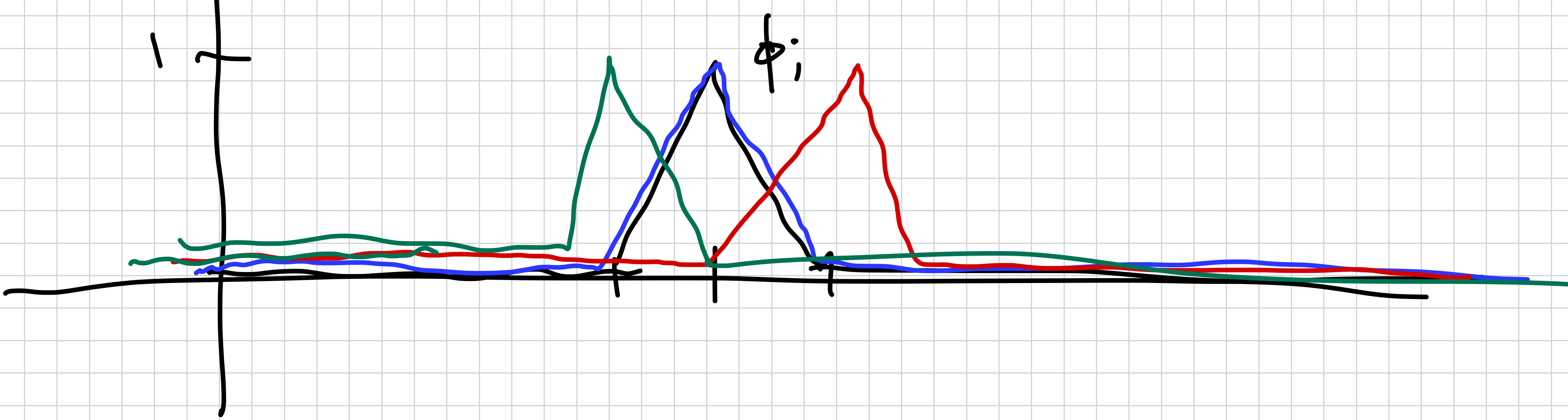
$$n \qquad \int_0^1 \sum_{i=0}^{n} u_i \phi_i' \phi_n' \, dx = \int_0^1 f \phi_n \, dx$$

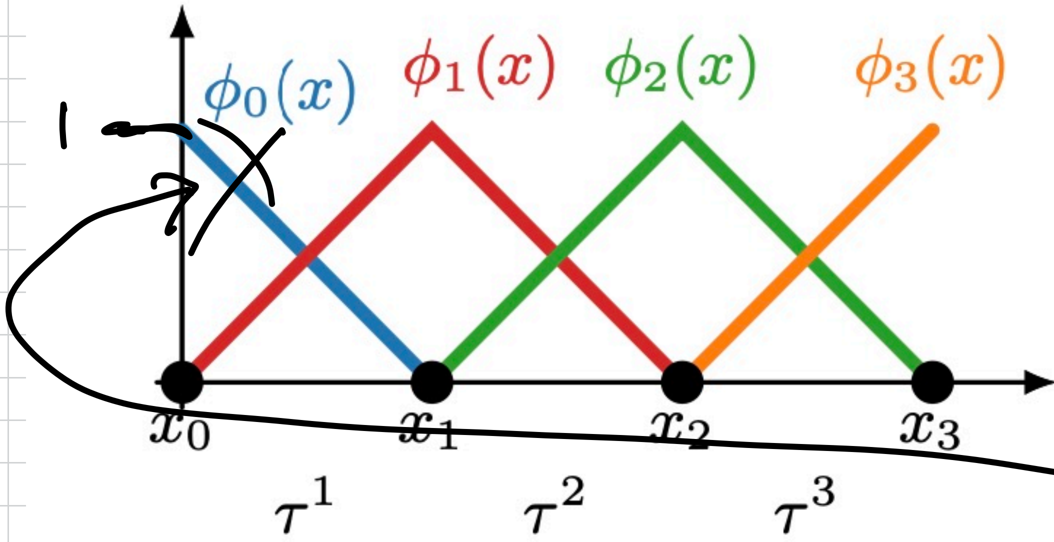
$$\langle \phi_8', \phi_1' \rangle$$

$$\begin{bmatrix} \langle \phi_0', \phi_0' \rangle & \langle \phi_1', \phi_0' \rangle & \cdots & \langle \phi_n', \phi_0' \rangle \\ \langle \phi_0', \phi_1' \rangle & \langle \phi_1', \phi_1' \rangle & \cdots & \langle \phi_n', \phi_1' \rangle \\ \vdots & & & \\ \langle \phi_0', \phi_n' \rangle & \langle \phi_1', \phi_n' \rangle & \cdots & \langle \phi_n', \phi_n' \rangle \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \langle f, \phi_0 \rangle \\ \langle f, \phi_1 \rangle \\ \\ \langle f, \phi_n \rangle \end{bmatrix}$$



$$\langle \phi'_8 , \phi'_i \rangle = ?\ \partial$$





$$u^h = \sum u_i \phi_i(x)$$

$$u^h(0) = 0$$

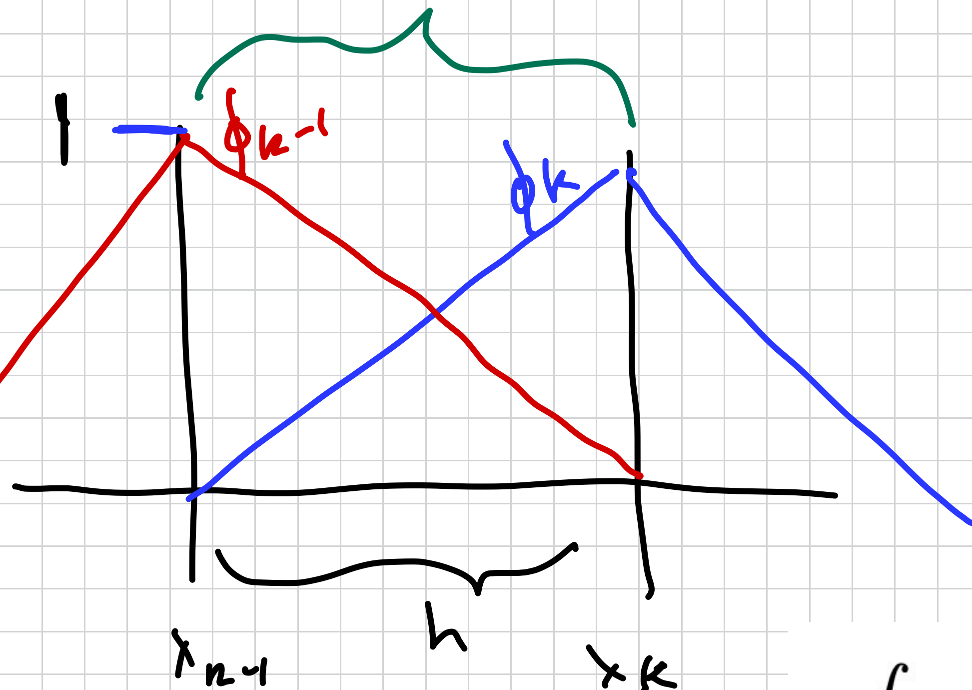
$$\Downarrow$$

$$u_0 = 0$$

$$= \sum_{i=1}^{3} u_i \phi_i(x)$$

$$\int_{x_{k-1}}^{x_k} \phi_{k-1}' \, \phi_{k-1}' = \int_{x_{k-1}}^{x_k} \left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right) dx$$

$$= \frac{1}{h}$$



$$\int_{x_{k-1}}^{x_k} \phi_{k-1}' \, \phi_k' \, dx = \int_{x_{k-1}}^{x_k} \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right)$$

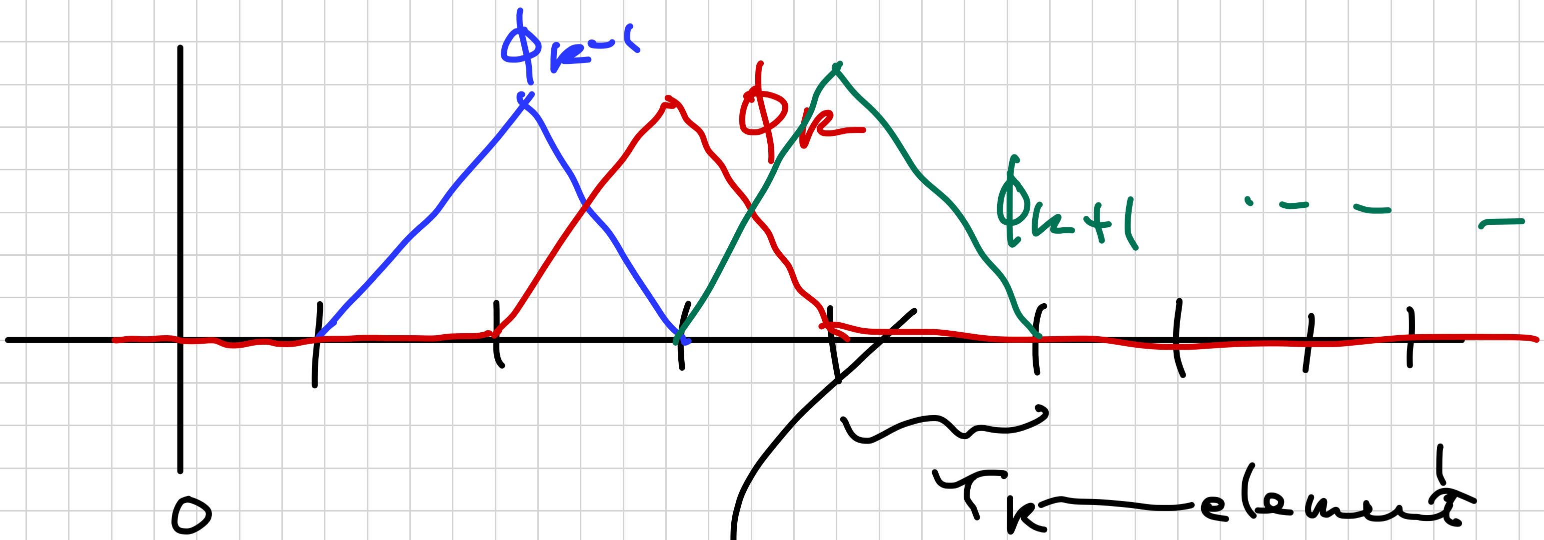
$$= -\frac{1}{h}$$

$$\int_{\tau^k} \phi_{k-1}'(x)\phi_{k-1}'(x)\,dx = \frac{1}{h},$$

$$\int_{\tau^k} \phi_{k-1}'(x)\phi_k'(x)\,dx = -\frac{1}{h}, \quad \text{and}$$

$$\int_{\tau^k} 1\,\phi_{k-1}(x)\,dx = \frac{h}{2},$$

use $f(x) \equiv 1$
as a start



A^k , f^k

$$f^k = \begin{bmatrix} \langle f, \phi_{k-1} \rangle \\ \langle f, \phi_k \rangle \end{bmatrix}$$

$$A^k = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad f^k = \frac{h}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{for all } k.$$

$$= \begin{bmatrix} \langle \phi'_{k-1}, \phi'_{k-1} \rangle & \langle \phi'_{k-1}, \phi'_k \rangle \\ \langle \phi'_k, \phi'_{k-1} \rangle & \langle \phi'_k, \phi'_k \rangle \end{bmatrix}$$

$$A = \frac{1}{h}\begin{bmatrix} 1 & -1 & & \\ -1 & 1 & & \\ & & & \\ & & & \end{bmatrix} + \frac{1}{h}\begin{bmatrix} & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & \end{bmatrix} + \frac{1}{h}\begin{bmatrix} & & & \\ & & & \\ & & 1 & -1 \\ & & -1 & 1 \end{bmatrix} \tag{4.98a}$$

$$= \frac{1}{h}\begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}. \tag{4.98b}$$

$$\begin{bmatrix} 1 & & & \\ & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ & 2 & -1 & \\ -1 & 2 & -1 & \\ & -1 & 1 & \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ rhs_1 \\ rhs_2 \\ rhs_3 \end{bmatrix}$$

$f = 1$

---

let $f = \frac{\pi^2}{4} \sin \frac{\pi x}{2}$

$\to \quad u_{exact} = \sin \frac{\pi x}{2}$

Need ① a test

② to compute

$\cancel{\langle f, \phi_0 \rangle}$

$\langle f, \phi_1 \rangle$

$\langle f, \phi_2 \rangle$

$\vdots$

$$= \int_0^1 f \, \phi_k \, dx$$

use 1pt midpt rule