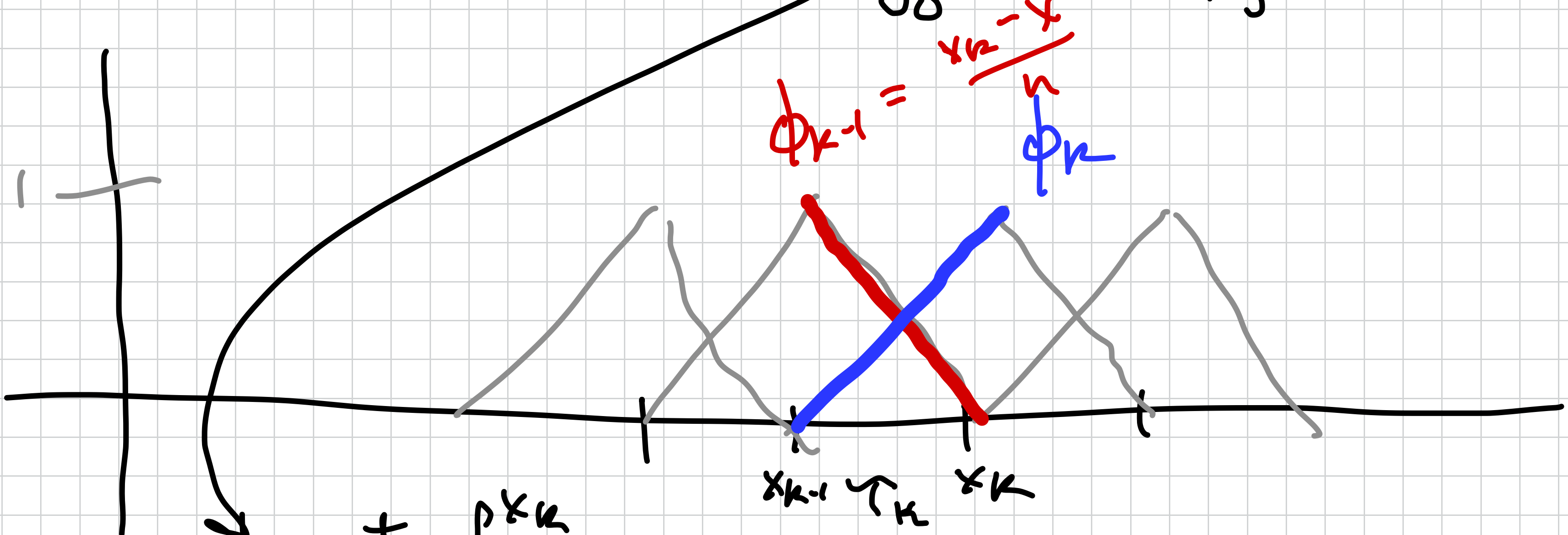
How to handle $\int_0^1 f(x)\,\phi_j(x)\,dx$



want $\int_{x_{k-1}}^{x_k} f(x)\cdot\phi_i\,dx$

change of variables: let $t = \frac{x - x_{k-1}}{h} \Rightarrow t = 0\ldots1$

$\Rightarrow$ $x = x_{k-1} + h t$

$dx = h\,dt$

$$\int_0^1 f(x_{k-1}+ht)\cdot\phi_{k-1}(x_{k-1}+ht)\cdot h\,dt$$

$$\phi_{k-1} = \frac{x_k - x}{h} = \frac{x_k - x_{k-1} - ht}{h}$$

$$= 1 - t$$

$$\int_0^1 f(x_{k-1} + ht) \cdot (1-t)\, h \; dt$$

midpoint rule (1pt Gauss Quadrature)

let $t_{mid} = 0.5$

width $= 1.0$

$$\approx f(x_{k-1} + h \cdot t_{mid}) \cdot (1 - t_{mid}) \cdot h$$

we have

$$-\partial_x\left(k(x)\, \partial_x u \right) = f$$

$$u(0) = \alpha$$
$$u'(1) = \beta$$

① $k(x) > 0$?

② $\alpha \neq 0$

③ $\beta \neq 0$

① $k(x) > 0$:

$$-\partial_x \left(k(x) \partial_x u \right) = f$$

$$\rightarrow \int_0^1 -\partial_x \left(k(x) \partial_x u \right) v \, dx = \int_0^1 f \, v \, dx$$

$$\text{I.B.P.} \rightarrow \underbrace{\int_0^1 k(x) \partial_x u \, \partial_x v}_{\text{how to compute?}} - \underset{\uparrow \text{ details}}{k(x) \partial_x u \, v \Big|_0^1} = \int_0^1 f v$$

want $\displaystyle\int_{x_{k-1}}^{x_k} k(x) \, \phi_i'(x) \, \phi_j'(x) \, dx$

$$\phi_i = \frac{x_k - x}{h} \text{ or } \frac{x - x_{k-1}}{h}$$

$$\rightarrow \int_{x_{k-1}}^{x_k} k(x) \cdot \left(\mp \frac{1}{h} \right) \left(\mp \frac{1}{h} \right) dx$$

$$\rightarrow \phi_i' = \mp \frac{1}{h}$$

change: $x = x_{k-1} + h t$

$dx = h \, dt$

$$\int_0^1 k(x_{k-1} + h t) \cdot \underbrace{\pm \frac{1}{h^2}}_{} \cdot h \, dt$$

② $\alpha \neq 0$

$$-u_{xx} = f$$
$$u(0) = \alpha$$
$$u'(0) = 0$$

find $u \in V$ s.t.

weak form:

$$\int_0^1 -u_{xx} v \, dx = \int_0^1 f v$$

$$\rightarrow \boxed{\int_0^1 u_x v_x \, dx - u_x v \Big|_0^1 = \int_0^1 f v \, dx} \quad \forall v \in V.$$

$$V = \left\{ v \in L^2 \;\middle|\; \begin{array}{l} a(v,v) < \infty \\ v(0) = 0 \end{array} \right\}$$

find $u \in V$ s.t.

$$a(u,v) = \langle f, v\rangle \quad \forall v \in V.$$

Another view: find u that minimizes

$$J(u) = \frac{1}{2} a(u,u) - \langle f, u\rangle$$

Is this the same?

$$J'(u)[v] = \lim_{\varepsilon \to 0} \frac{J(u+\varepsilon v) - J(u)}{\varepsilon}$$

"derivative J in the direction v"

$$= \lim_{\varepsilon \to 0} \frac{\frac{1}{2} a(u+\varepsilon u, u+\varepsilon u) - \langle f, u+\varepsilon u\rangle - \frac{1}{2} a(u,u) + \langle f, u\rangle}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\cancel{\frac{1}{2} a(u,u)} + \varepsilon a(u,u) + \frac{1}{2}\varepsilon^2 a(u,u) - \cancel{\frac{1}{2} a(u,u} - \varepsilon \langle f, u\rangle}{\varepsilon}$$

$$= a(u,u) - \langle f, u\rangle$$

u satisfies

$$J(u) \le J(u+\varepsilon v) \quad \forall \; u+\varepsilon v \in \{ v \mid v(0)=\alpha \}$$

Take away:

Find $u \in \{ v \in L^2 \mid a(u,u) < \infty,\ u(0) = \alpha \}$

$$\int_0^1 u_x v_x \, dx = \int_0^1 f v \, dx$$

for all $v \in \{ v \in L^2 \mid a(v,v) < \infty,\ v(0) = 0 \}$

- Alternative view:

$$-u_{xx} = f$$

$$u(0) = \alpha$$

$$u'(1) = \beta$$

let $u_0(x) \in C^2$

let $u_0(0) = \alpha$

Now let $w(x) = u(x) - u_0(x)$

$$\rightarrow \quad -w_{xx} = f + (u_0(x))_{xx}$$

$$w(0) = 0$$

③ $\beta \neq 0$

$$-u_{xx} = f$$
$$u(0) = 0$$
$$u'(1) = \beta$$

$$\rightarrow \int_0^1 -u_{xx}\, v = \int_0^1 f\, v$$

$$\rightarrow \int_0^1 u_x v_x\, dx - u_x v\Big|_0^{1} = \int_0^1 f\, v\, dx$$

$$\int_0^1 u_x v_x\, dx = \int_0^1 f\, v\, dx + \beta \cdot v(1)$$

$$-u_x(1)v(1)$$

$u^h = \sum u_i \phi_i$

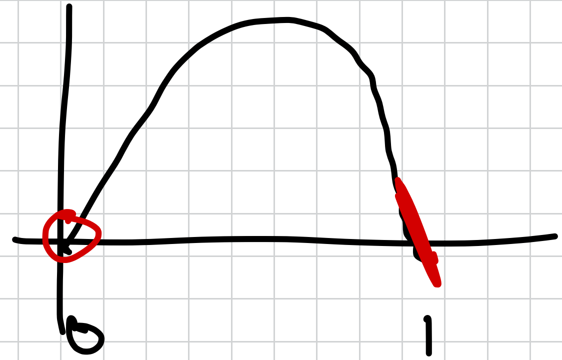
find u^h s.t.

$$\int_0^1 u^h_x \cdot (\phi_j)_x\, dx = \int_0^1 f\, \phi_j\, dx + \beta \cdot \phi_j(1)$$

$$\forall \phi_j$$

$$-u_{xx} = f$$

let $\boxed{u(x) = \sin(\pi x)}$



$$\Rightarrow u'(x) = \pi \cos \pi x$$

$$\Rightarrow u'(1) = -\pi$$

$$u(0) = 0$$

$$\begin{cases} -u_{xx} = \pi^2 \sin \pi x \\ u(0) = 0 \\ u'(1) = -\pi \end{cases}$$

recommendation:

① β
② kappa
③ α

$$\underbrace{\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 2 & -1 & & \\ \vdots & -1 & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \ddots \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_n \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} \alpha \\ f_1 \\ \vdots \\ \vdots \\ f_n \end{bmatrix}}_{f}$$

$u = \cos \pi x \quad \rightarrow \quad -u_{xx} = \pi^2 \cos \pi x$

$u(0) = 1$

$u'(1) = 0$