How to handle \( \int_0^1 f(x) \phi_j(x) \, dx \)

\[ \phi_{k-1} = \frac{x_{k-1} - x}{h} \quad \phi_k = \frac{x - x_{k-1}}{h} \]

\[ \frac{x_k - x_{k-1} - h t}{h} = 1 - t \]

\[ \text{change of variables:} \quad \text{let} \quad t = \frac{x - x_{k-1}}{h} \Rightarrow t = 0 \ldots 1 \]

\[ x = x_{k-1} + h t \]

\[ dx = h \, dt \]

\[ \int_0^1 f(x_{k-1} + h t) \cdot \phi_{k-1}(x_{k-1} + h t) \cdot h \, dt \]
\[ \int_0^1 f(x_{n-1} + h \cdot t_{\text{mid}}) \cdot (1-t) \cdot h \cdot dt \]

midpoint rule (1pt Gauss Quadrature)

let \( t_{\text{mid}} = 0.5 \)

width = 1.0

\[ f(x_{n-1} + h \cdot t_{\text{mid}}) \cdot (1-t_{\text{mid}}) \cdot h \]
we have
\[- \partial_x \left( k(x) \partial_x u \right) = f \]
\[u(0) = \alpha\]
\[u'(1) = \beta\]

① \( k(x) > 0 \) ?
② \( \alpha \neq 0 \)
③ \( \beta \neq 0 \)
\( k(x) > 0:\)
\[
- \frac{\partial}{\partial x} (k(x) \partial_x u) = f
\]
\[
\Rightarrow \int_0^1 - \frac{\partial}{\partial x} (k(x) \partial_x u) \, v \, dx = \int_0^1 f \, v \, dx
\]
IBP.
\[
\Rightarrow \int_0^1 k(x) \partial_x u \, \partial_x v - k(x) \partial_x u \, v \bigg|_0^1 = \int_0^1 f \, v
\]
\( \text{how to compute?} \)

want
\[
\int_{x_{n-1}}^{x_n} k(x) \phi'_i(x) \phi'_j(x) \, dx
\]
\[
\phi_i = \frac{x_k - x}{x_k - x_{k-1}}
\]
\[
\Rightarrow \phi'_i = \frac{1}{h}
\]

change:
\[
x = x_{n-1} + h \, t
\]
\[
dx = h \, dt
\]
\[
\int_0^1 k(x_{n-1} + h \, t) \cdot \pm \frac{1}{h^2} \cdot h \, dt
\]
\( x \to 0 \)

\[-u_{xx} = f \]

\[u(0) = x \]

\[u'(0) = 0 \]

\[
\text{weak form:}
\int_0^1 -u_{xx} v \, dx = \int_0^1 f v \, dx
\]

\[
\int_0^1 u_x v_{xx} \, dx - u_{xx} v \bigg|_0^1 = \int_0^1 f v \, dx
\]

\[
V = \left\{ v \in L^2 \mid a(v,v) > 0 \right\}
\]

\[v(0) = 0 \]
To find $u \in V$ such that

$$a(u, v) = \langle f, v \rangle \quad \forall v \in V.$$ 

Another view: find $u$ that minimizes

$$J(u) = \frac{1}{2} a(u, u) - \langle f, u \rangle$$

Is this the same?

$$J'(u)[v] = \lim_{\varepsilon \to 0} \frac{J(u + \varepsilon v) - J(u)}{\varepsilon}$$

"Derivative of $J$ in the direction $v$:"

$$= \lim_{\varepsilon \to 0} \frac{1}{2} a(u + \varepsilon v, u + \varepsilon v) - \langle f, u + \varepsilon v \rangle$$

$$- \frac{1}{2} a(u, u) + \langle f, u \rangle$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2} a(u, u) + \varepsilon a(u, v) + \frac{1}{2} \varepsilon^2 a(v, v) - \frac{1}{2} a(u, u) - \langle f, u \rangle$$

$$= a(u, u) - \langle f, u \rangle$$
\[ u \text{ satisfies } \]
\[ J(u) \leq J(u + \varepsilon v) + \varepsilon v + \frac{v^2}{2\varepsilon} \]
\[ \forall \varepsilon \in \mathbb{R} \quad v(0) = \alpha^3 \]

Take away:

Find \( u \in \{ v \in L^2 | a(u, v) < \infty \} \)

\( u(0) = \alpha^3 \)

\( \int_0^1 u_x v_x \, dx = \int_0^1 f \, v \, dx \)

for all \( v \in \{ v \in L^2 | a(v, v) < \infty \} \quad v(0) = 0 \)
Alternative view:

\[-u_{xx} = f\]

\[u(0) = \alpha\]
\[u'(0) = \beta\]

Let \(u_0(x) \in C^2\)

Let \(u_0(0) = \alpha\)

Now let \(w(x) = u(x) - u_0(x)\)

\[\Rightarrow -w_{xx} = f + (u_0(x))_{xx}\]

\(w(0) = 0\)
\[ u = \text{int}_\phi \left( \int_0^1 u \, dx \right) \]

\[ \int_0^1 u \, dx = \int_0^1 \int_0^1 u \, dy \, dx \]

\[ \int_0^1 \int_0^1 u \, dy \, dx = 0 \]

\[ \int_0^1 \int_0^1 \phi \, dy \, dx = 0 \]

\[ \phi \neq 0 \]

\[ \int_0^1 \int_0^1 \phi \, dy \, dx = \int_0^1 \int_0^1 \phi \, dy \, dx + B \cdot v(c) \]

\[ \int_0^1 \int_0^1 \phi \, dy \, dx - \int_0^1 \int_0^1 \phi \, dy \, dx = 0 \]

\[ \int_0^1 \int_0^1 \phi \, dy \, dx = B \cdot v(c) \]

\[ \int_0^1 \int_0^1 \phi \, dy \, dx = B \cdot v(c) \]

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\[ \int_0^1 \int_0^1 \phi \, dy \, dx = 0 \]

\[ \int_0^1 \int_0^1 \phi \, dy \, dx = \int_0^1 \int_0^1 \phi \, dy \, dx + B \cdot v(c) \]

\[ \int_0^1 \int_0^1 \phi \, dy \, dx = 0 \]

\[ \int_0^1 \int_0^1 \phi \, dy \, dx = \int_0^1 \int_0^1 \phi \, dy \, dx + B \cdot v(c) \]

\[ \int_0^1 \int_0^1 \phi \, dy \, dx = 0 \]
\[-u_{xx} = f\]

Let \( u(x) = \sin(\pi x) \)

\[
\begin{align*}
\Rightarrow u'(x) &= \pi \cos \pi x \\
\Rightarrow u'(1) &= -\pi \\
\quad u(0) &= 0
\end{align*}
\]

\[
\begin{align*}
\int -u_{xx} &= \pi^2 \sin \pi x \\
\quad u(0) &= 0 \\
\quad u'(1) &= -\pi
\end{align*}
\]

Recommendation:

1. \( p^2 \)  
2. \( k \)  
3. \( \alpha \)
\( u = \cos \pi x \quad \Rightarrow \quad -u_{xx} = \pi^2 \cos \pi x \\
\quad u(0) = 1 \\
\quad u'(1) = 0 \\
\quad u'(0) = 0 \)