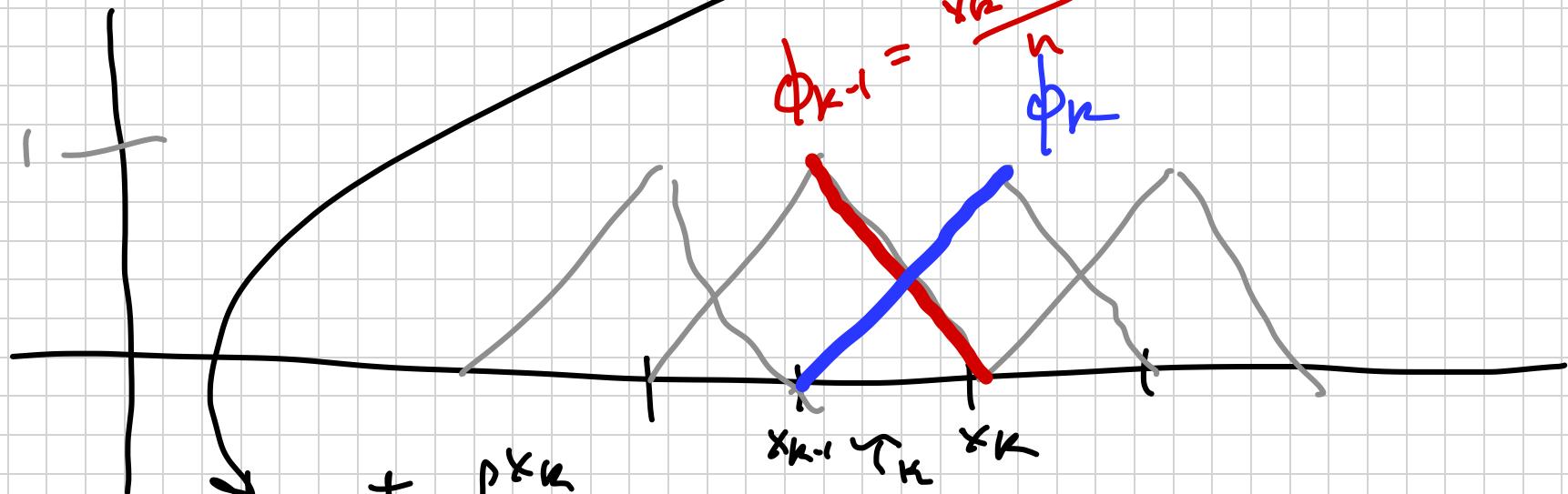


How to handle $\int_0^1 f(x) \phi_i(x) dx$



hand $\int_{x_{k-1}}^{x_k} f(x) \cdot \phi_i dx$

change of variables: let $t = \frac{x - x_{k-1}}{h} \Rightarrow t = 0 \dots 1$
 $\Rightarrow x = x_{k-1} + ht$
 $dx = h dt$

$\Rightarrow \int_0^1 f(x_{k-1} + ht) \cdot \phi_{k-1}(x_{k-1} + ht) \cdot h dt$

$$\begin{aligned}\phi_{k-1} &= \frac{x_k - x}{h} = \frac{x_k - x_{k-1} - ht}{h} \\ &= 1 - t\end{aligned}$$

$$\int_0^1 f(x_{k-t} + ht) \cdot (1-t) h dt$$

mid point rule (1 pt Gauss Quadrature)

let $t_{\text{mid}} = 0.5$

width = 1.0

$$\approx f(x_{k-1} + h \cdot t_{\text{mid}}) \cdot (1 - t_{\text{mid}}) \cdot h$$

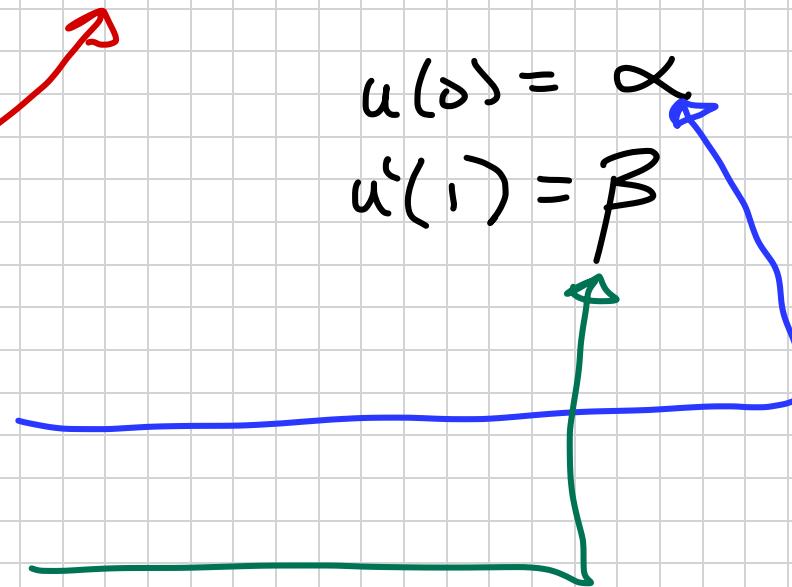
We have

$$-\partial_x(K(x) \partial_x u) = f$$

① $K(x) > 0$?

② $\alpha \neq 0$

③ $\beta \neq 0$



① $k(x) > 0$:

$$-\partial_x(k(x)\partial_x u) = f$$

$$\Rightarrow \int_0^1 -\partial_x(k(x)\partial_x u) v \, dx = \int_0^1 f v \, dx$$

I.B.P. \rightarrow

$$\int_0^1 k(x)\partial_x u \partial_x v - k(x)\partial_x u v \Big|_0^1 = \int_0^1 f v$$

how to compute?

details

want

$$\int_{x_{n-1}}^{x_n} k(x) \phi_i'(x) \phi_j'(x) \, dx$$

$$\phi_i' = \frac{x_k - x}{h}$$

$$\text{or}$$

$$\frac{x - x_{k-1}}{h}$$

$$\rightarrow \int_{x_{n-1}}^{x_n} k(x) \cdot \left(\pm \frac{1}{h}\right) \left(\mp \frac{1}{h}\right) \, dx$$

$$\rightarrow \phi_i'' = \mp \frac{1}{h}$$

change:

$$x = x_{n-1} + h t$$

$$dx = h dt$$

$$\int_0^1 k(x_{n-1} + h t) \cdot \underbrace{\mp \frac{1}{h^2}}_{-} \cdot h dt$$

② $\alpha \neq 0$

$$-u_{xx} = f$$

$$u(0) = \alpha$$

$$u'(0) = 0$$

find $u \in V$ st.

weak form:

$$\int_0^1 -u_{xx} v \, dx = \int_0^1 f v \, dx$$
$$\rightarrow \left[\int_0^1 u_x v_x \, dx - u_x v \Big|_0^1 \right] + \boxed{\int_0^1 f v \, dx} = 0 \quad \forall v \in V.$$

$$V = \left\{ v \in L^2 \mid \begin{array}{l} a(v, v) < \infty \\ v(0) = 0 \end{array} \right\}$$

Find $u \in V$ s.t.

$$a(u, v) = \langle f, v \rangle \quad \forall v \in V.$$

Another view: find u that minimizes

$$J(u) = \frac{1}{2} a(u, u) - \langle f, u \rangle$$

Is this the same?

$$J'(u)[v] = \lim_{\epsilon \rightarrow 0} \frac{J(u + \epsilon v) - J(u)}{\epsilon}$$

"derivative J in the direction $v"$

$$= \lim_{\epsilon \rightarrow 0} \frac{\frac{1}{2} a(u + \epsilon v, u + \epsilon v) - \langle f, u + \epsilon v \rangle}{\epsilon}$$
$$= \frac{\frac{1}{2} a(u, u) + \epsilon a(u, v) + \frac{1}{2} \epsilon^2 a(v, v) - \langle f, u \rangle}{\epsilon}$$

$$= \lim_{\epsilon \rightarrow 0} \frac{\frac{1}{2} a(u, u) + \epsilon a(u, v) + \frac{1}{2} \epsilon^2 a(v, v) - \langle f, u \rangle}{\epsilon}$$

$$= a(u, v) - \langle f, u \rangle$$

u satisfies

$$J(u) \leq J(u + \varepsilon v) \quad \forall u + \varepsilon v \in \{v \mid v(0) = \alpha\}$$

Take away:

$$\text{Find } u \in \{v \in L^2 \mid a(u, v) < \infty, u(0) = \alpha\}$$

$$\int_0^1 u_x v_x dx = \int_0^1 f v dx$$

$$\text{for all } v \in \{v \in L^2 \mid a(v, v) < \infty, v(0) = 0\}$$

Alternative view:

$$- u_{xx} = f$$

$$u(0) = \alpha$$

$$u'(1) = \beta$$

let $u_o(x) \in C^2$

let $u_o(0) = \alpha$

Now let $w(x) = u(x) - u_o(x)$

$$\rightarrow -w_{xx} = f + (u_o(x))_{xx}$$

$$w(0) = 0$$

③ $\beta \neq 0$

$$-u_{xx} = f$$

$$k(\Delta) = 0 \quad \leftarrow$$

$$u'(c) = \beta$$

$$\rightarrow \int_0^1 -u_{xx} v \, dx = \int_0^1 f v \, dx$$

$$\rightarrow \int_0^1 u_x v_x \, dx - u_x v \Big|_0^1 = \int_0^1 f v \, dx$$

$$\int_0^1 u_x v_x \, dx = \int_0^1 f v \, dx + \beta \cdot v(1)$$

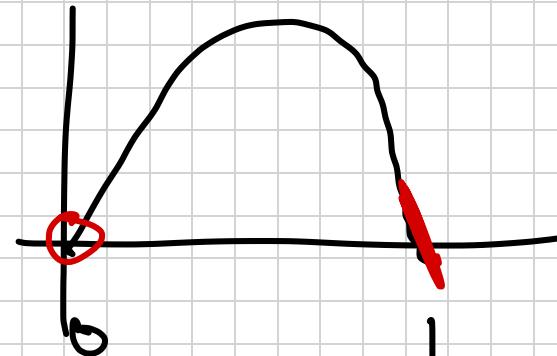
$$-u_x(1)v(1)$$

$$u_h^n = \sum u_i \phi_i$$

$$\text{find } u_h^n \text{ s.t. } \int_0^1 u_x^n \cdot (\phi_j)_x \, dx = \int_0^1 f \phi_j \, dx + \beta \cdot \phi_j(1) \quad \forall \phi_j$$

$$-u_{xx} = f$$

let $u(x) \approx \sin(\pi x)$



$$\Rightarrow u'(x) = \pi \cos \pi x$$

$$\Rightarrow u'(1) = -\pi$$

$$u(0) = 0$$

$$\left. \begin{array}{l} -u_{xx} = \pi^2 \sin \pi x \\ u(0) = 0 \end{array} \right.$$

$$u'(1) = -\pi$$

- recommendations:
- ① β
 - ② $k \rightarrow \infty$
 - ③ α

$$\begin{bmatrix}
 1 & 0 & 0 \\
 0 & 2 & -1 \\
 0 & -1 & -1
 \end{bmatrix} A =
 \begin{bmatrix}
 0 & u_0 & u_1 & \dots & u_n \\
 u_0 & u_1 & \vdots & \vdots & u_n \\
 u_1 & \vdots & \ddots & \vdots & u_n \\
 \vdots & \vdots & \vdots & \ddots & u_n \\
 u_n & u_n & u_n & \dots & u_n
 \end{bmatrix} = f$$

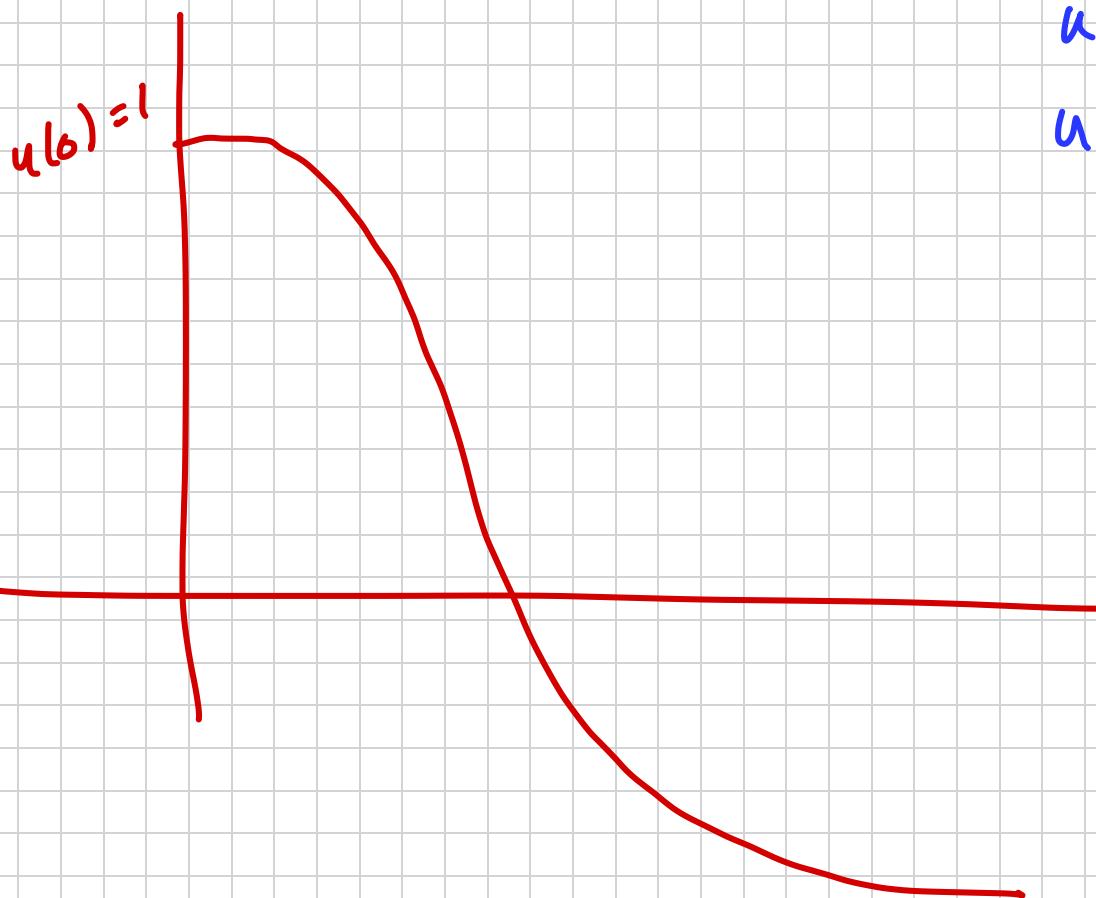
The diagram illustrates the matrix multiplication Au . The matrix A is a 3x3 matrix with entries 1, 0, 0; 0, 2, -1; and 0, -1, -1. The vector u is represented as a column vector $[u_0, u_1, \dots, u_n]^T$. The result of the multiplication is a vector f , also represented as a column vector $[f_0, f_1, \dots, f_n]^T$. Red brackets highlight the first row of A and the first element of u in the intermediate step, and red arrows point from the first row of A to the first element of f .

$$u = \cos \pi x \rightarrow$$

$$-u_{xx} = \pi^2 \cos \pi x$$

$$u(0) = 1$$

$$u'(1) = 0$$



$$u'(0) = 0$$