Today 3/25

Find $u \in \overset{\text{“some space”}}{V}$ s.t.

$$a(u,v) = \langle f, v \rangle \quad \forall\, v \in V$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\int_\Omega \nabla u \cdot \nabla v \, d\underline{x} \qquad\qquad \int_\Omega f v \, d\underline{x}$$

Function Spaces

$\boxed{C^m}$

connected
+

Let Ω be an open, bounded domain

$(0,1)$

$\overline{(0,1)} = [0,1]$

$C^0(\Omega)$ = all continuous fcns.

$$\to C^m(\Omega) = \left\{ f \in C^0(\Omega) \;\middle|\; \begin{array}{l} D^{\underline{k}} f \in C^0(\Omega) \\ |k| \le m \end{array} \right\}$$

$$\frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}}$$

$$|k| = k_1 + k_2 + k_3 \cdots k_n$$

closed + bounded
↓

$$C_0^m(\Omega) = \{ f \in C^m(\Omega) \mid f \text{ has compact support} \}$$

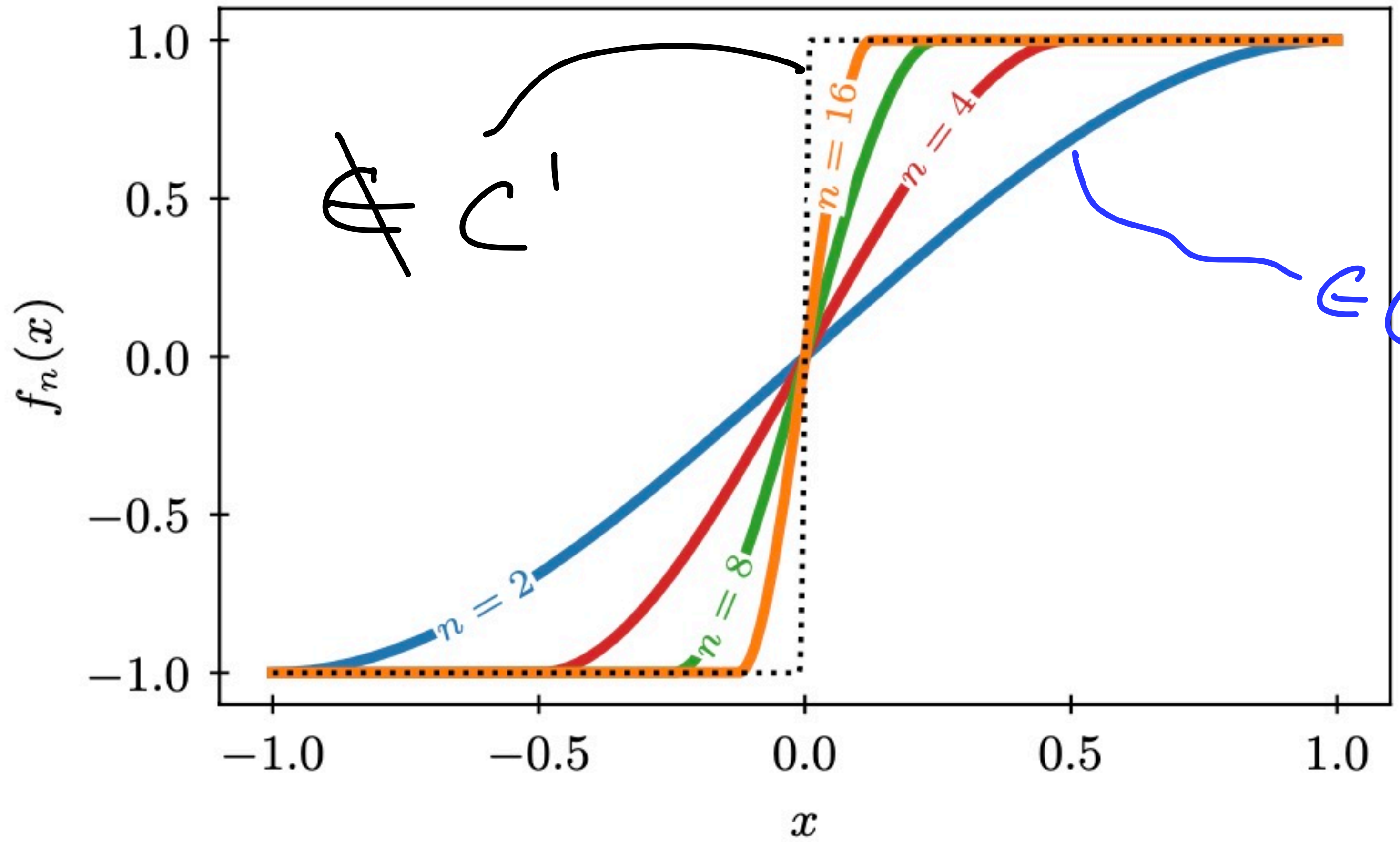
$$\operatorname{supp}(f) = \{ x \in \Omega \mid f(x) \neq 0 \}$$

Terminology

- let V be a vector space
  $f \in V$, $g \in V$ then $f+g \in V$
  $f \in V$, then $\alpha \cdot f \in V$ $(\alpha \in \mathbb{R})$
- add a norm to V. $\rightarrow$ $(V, \|\cdot\|)$
  "normed Vector space"
- Suppose that every Cauchy sequence has a limit f and $f \in V$.
  $\rightarrow$ "complete normed Vector space"
  "Banach Space"

Cauchy: $\lim_{m,n \to \infty} \|f_m - f_n\| = 0$

$$f_n(x) = \begin{cases} -1 & \text{if } x \leq -\frac{1}{n} \\ \frac{3n}{2}x - \frac{n^3}{2}x^3 & \text{if } x \in \left(-\frac{1}{n}, \frac{1}{n}\right) \\ 1 & x \geq \frac{1}{n} \end{cases}$$



$\notin C^1$

$\in C^1$

$$L^p = \left\{ u \;\middle|\; \int_\Omega |u|^p dx < \infty \right\}$$

with norm

$$\|u\|_p = \left(\int_\Omega |u|^p dx \right)^{1/p}$$

- V, vector space
- add an inner product $\langle u,v\rangle$

  ↳ this "induces" a norm

  $$\|u\| = \sqrt{\langle u,u\rangle}$$

  $$\langle \cdot,\cdot\rangle : V\times V \longrightarrow \mathbb{R}$$

  ① $\langle \alpha f+g,h\rangle = \alpha\langle f,h\rangle + \langle g,h\rangle$

  ② $\langle u,v\rangle = \langle v,u\rangle$

  ③ $\langle w,w\rangle \geq 0$

  $\langle w,w\rangle = 0$ iff $w=0$

- Suppose V is complete

  "Hilbert Space"

Case 1 $C^m(\bar{\Omega})$ with $\|\cdot\|_\infty$

$\uparrow$
max norm

- not Hilbert
  (no inner product)

Case 2 $C^n(\bar{\Omega})$ with $\|\cdot\|_2$

$$\langle u,v\rangle = \int_\Omega uv\,dx \Rightarrow \|u\|_2 = \left(\int_\Omega u^2 dx\right)^{1/2}$$

not complete.

Case 3

C_0^∞ + the limits of things in L^2-norm

+ $\|\cdot\|_2$-norm

→ Hilbert

$= L^2 = \{ u \mid \|\cdot\|_2 < \infty \}$

$$-u_{xx} = f$$

$$u(0) = u(1) = 0$$

find $u \in V$ s.t.

$$\int_{(0,1)} \underbrace{u_x}_{\text{is a weak derivative,}} v_x \, dx = \int_{(0,1)} f v \, dx \quad \forall v \in V.$$

The weak derivative

$$L^1(\Omega) = \{ u \mid \|u\|_1 < \infty \}$$

$$= \{ u \mid \int |u| \, dx < \infty \}$$

$$L^1_{loc}(\Omega) = \{ u \mid \int |u\phi| \, dx < \infty \text{ for all } \phi \in C_0^\infty(\Omega) \}$$

$$L^1 \subset L^1_{loc}$$

let $u \in L^1_{loc}(\Omega)$.

Then v is the weak derivative of $u \in L^1_{loc}$ if

$$\int_\Omega v \phi \, dx = - \int_\Omega u \frac{\partial \phi}{\partial x} \, dx$$

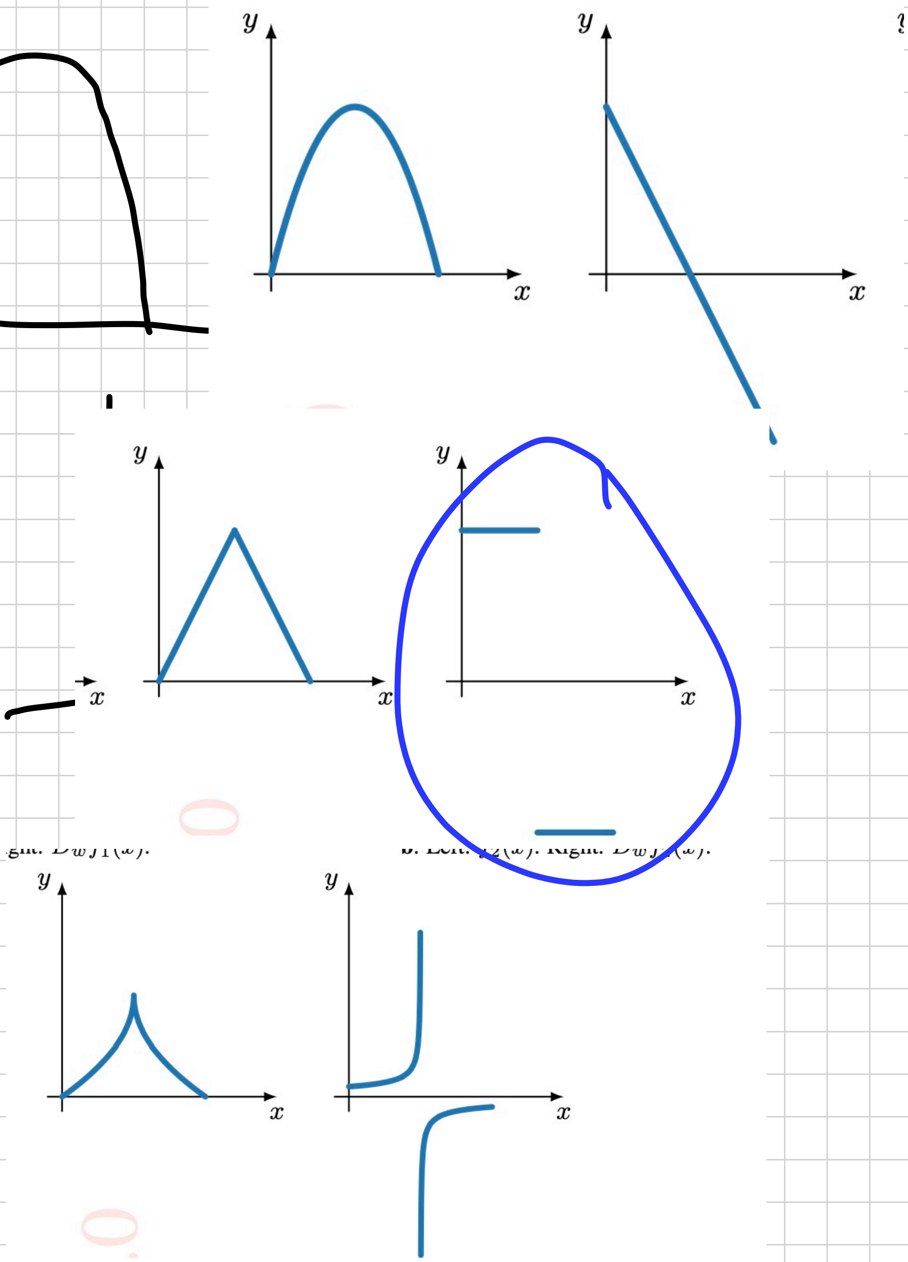
$$\forall \phi \in C_0^\infty(\Omega)$$

$$\text{“} \int_\Omega \frac{du}{dx} \phi \, dx \overset{I.B.P}{=} - \int_\Omega u \frac{d\phi}{dx} \, dx + \underbrace{u\phi \Big|_0^1}_{=0} \quad \forall \phi \text{”}$$

$$f_1(x) = 4(1-x)x$$

$$f_2(x) = \begin{cases} 2x & x \in [0, 1/2] \\ 2-2x & x \in [1/2, 1) \end{cases}$$

$$f_3(x) = \begin{cases} -\sqrt{\frac{1}{2} - x} + \sqrt{1/2} \\ -\sqrt{x - 1/2} + \sqrt{1/2} \end{cases}$$



c. Left: $f_3(x)$. Right: $D_w f_3(x)$.

Spaces with weak derivatives

$$H^1 = V = \{ v \in L^2 \mid D_w v \in L^2 \}$$

weak derivative.

$$H^0 = L^2$$

$$\downarrow$$

$$W^{k,p} = \{ u \mid \|u\|_{k,p} < \infty \}$$

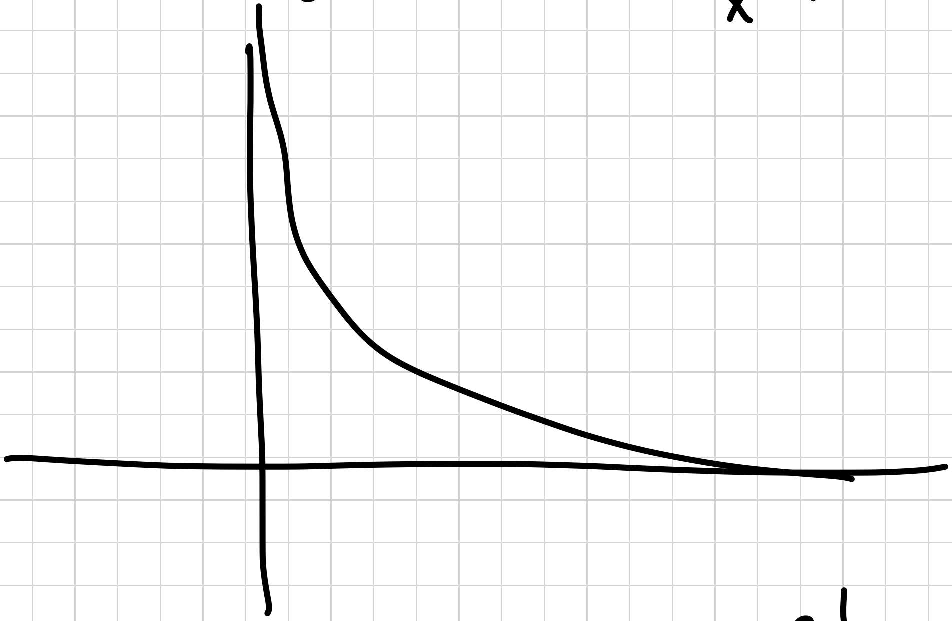
$$\|u\|_{k,p} = \Big(\sum_{|\alpha| \leq k} \|D_w^{\underline{\alpha}} u\|_p \Big)^{1/p}$$

if $p=2$

$$H^k = W^{k,2}$$

$\rightarrow$ Sobolev Spaces

try $u(x) = \frac{1}{x^{1/4}}$ on $(0,1)$



$$\int_0^1 |x^{-1/4}|^2 \, dx = \int_0^1 x^{-1/2} \, dx$$

$$= 2x^{1/2} \Big|_0^1$$

$$= 2$$

Things in L^2

if f is bounded on $(0,1)$
then $f \in L^2$:

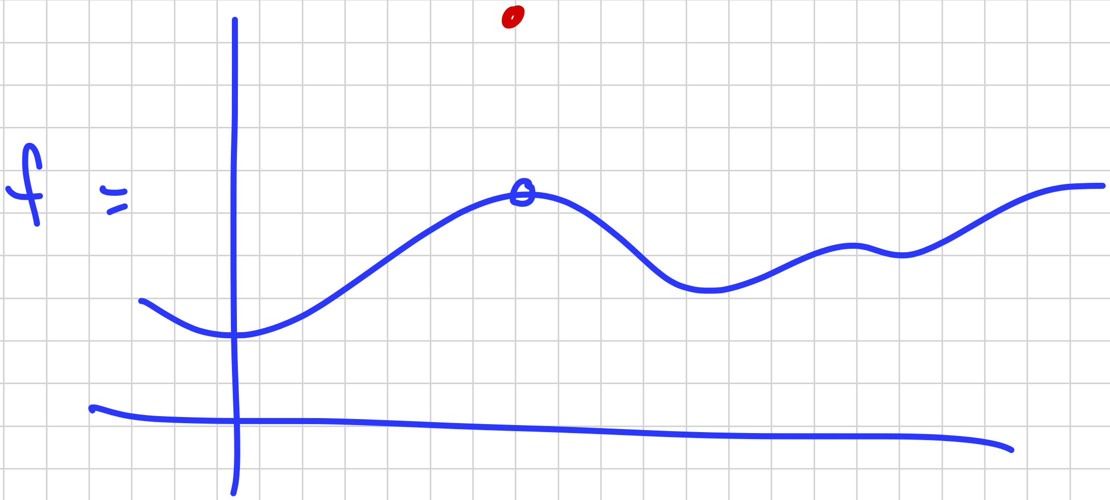
Proof

let $|f| < M$

$$\int_0^1 |f|^2 \, dx \le \int_0^1 M^2 \, dx$$

$$= M^2 < \infty$$

$f =$



$$H^1 = \{ u \in L^2 \mid D_w u \in L^2 \}$$

<u>1D</u>

$$= \{ u \in L^2 \mid u_x \in L^2 \}$$

$$\|u\|_{H^1}^2 = \|u\|_2^2 + \underset{\langle u_x , u_x \rangle}{\underset{\shortparallel}{\|u_x\|_2^2}}$$