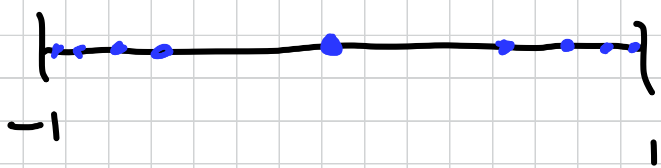
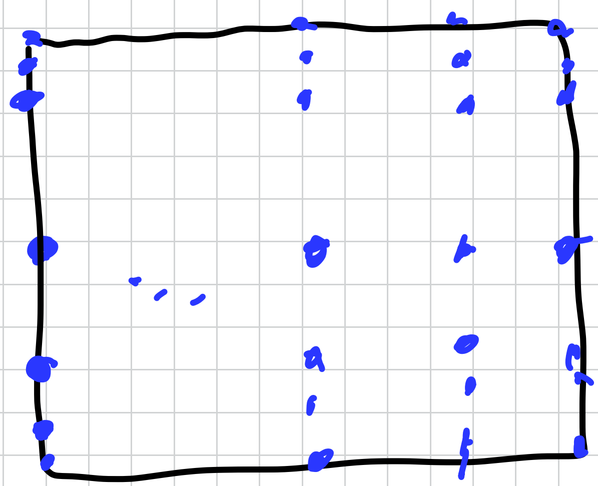
$$\int g(x,y)\,dx \simeq \sum_{i=1} w_i f(x_i, y_i)$$

1D



2D



2D

1D

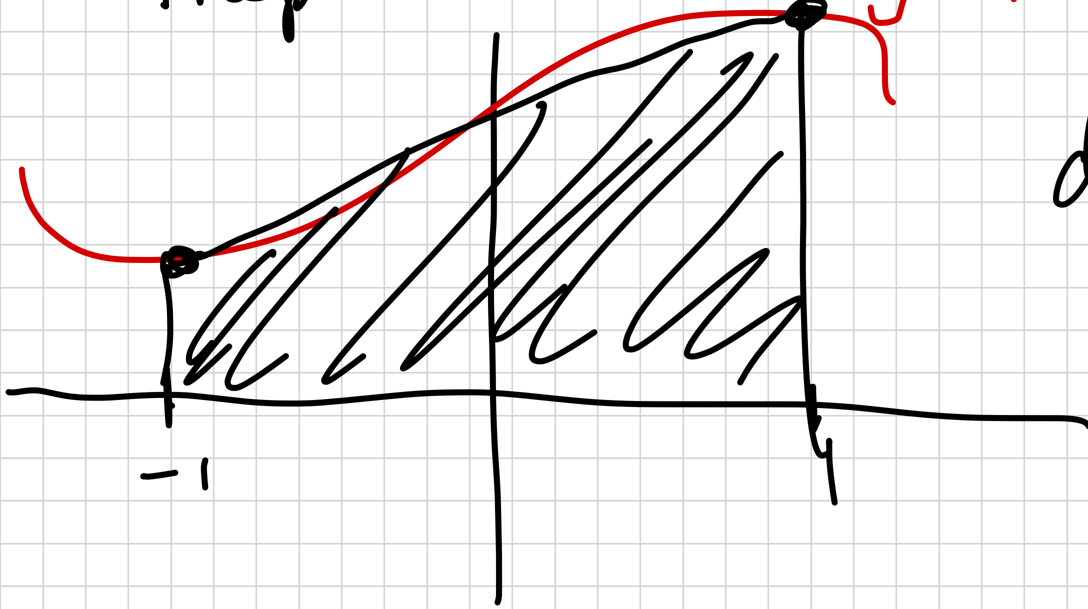
$$\int_{-1}^{1} g(x)\,dx \simeq \sum_{i=0}^{m} w_i\, f(x_i)$$

weights nodes

example

$$= 2 \cdot f(0) \quad \text{(midpoint)}$$

1D Trapezoid:



degree of precision:

What about Gauss Quadrature?

e.g. 3pt. Gauss Quad:

$$\int_{-1}^{1} g(x)\,dx \approx w_0 g(x_0) + w_1 g(x_1) + w_2 g(x_2)$$

unknowns

want to hold for

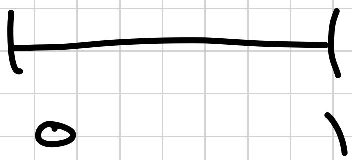
$g(x) = 1$
$g(x) = x$
$g(x) = x^2$
$g(x) = x^3$
$g(x) = x^4$
$g(x) = x^5$

constraints

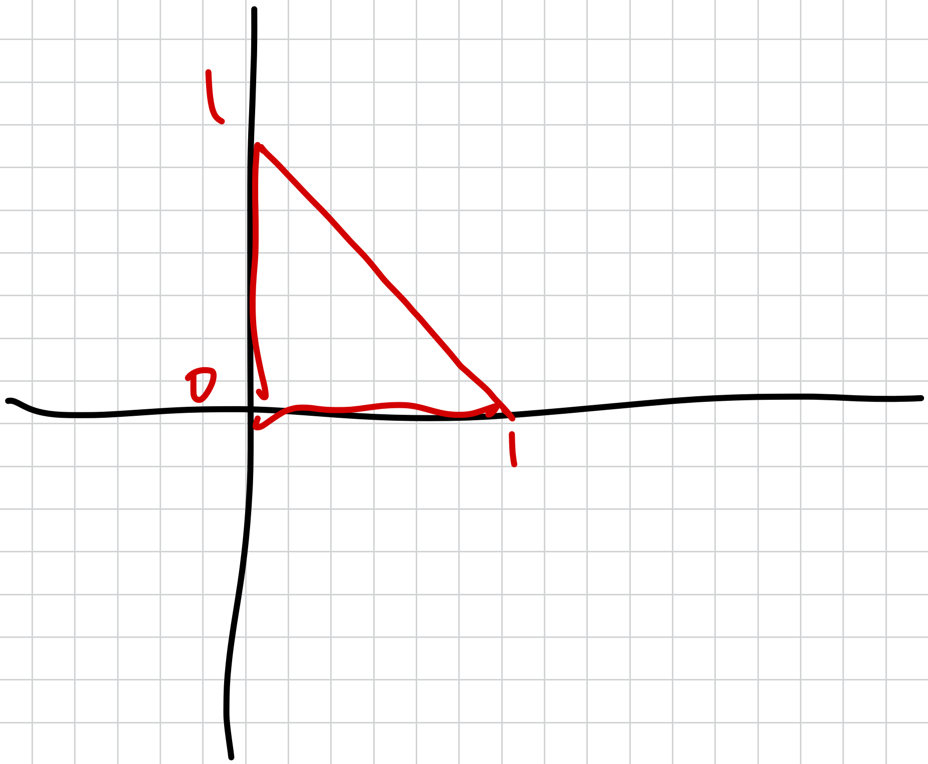
$g(x)=1$: $2 = \int_{-1}^{1} 1\,dx = w_0 \cdot 1 + w_1 \cdot 1 + w_2 \cdot 1$

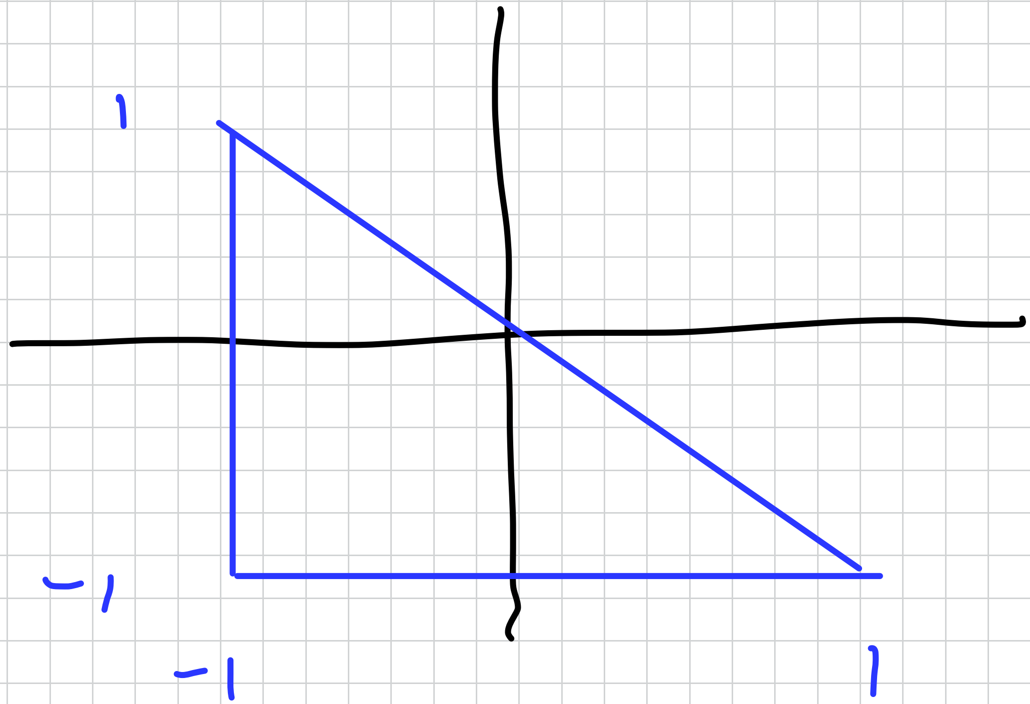
$g(x)=x$: $0 = \int_{-1}^{1} x\,dx = w_0 x_0 + w_1 x_1 + w_2 x_2$

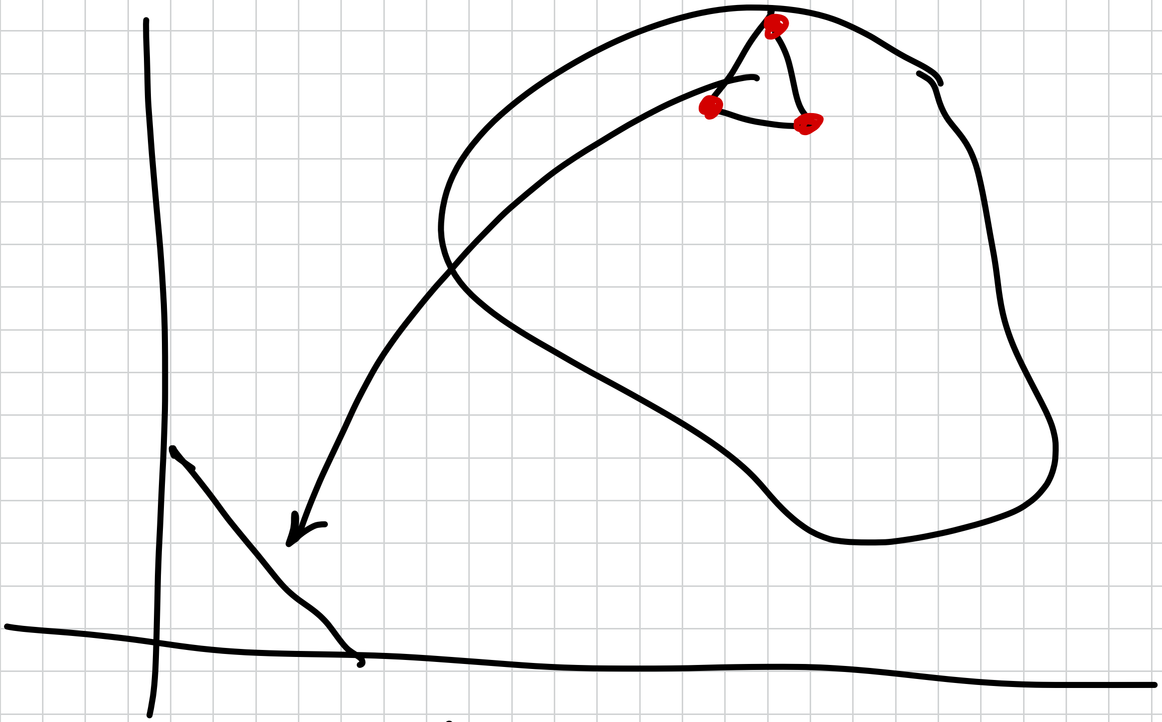
$\vdots$



or

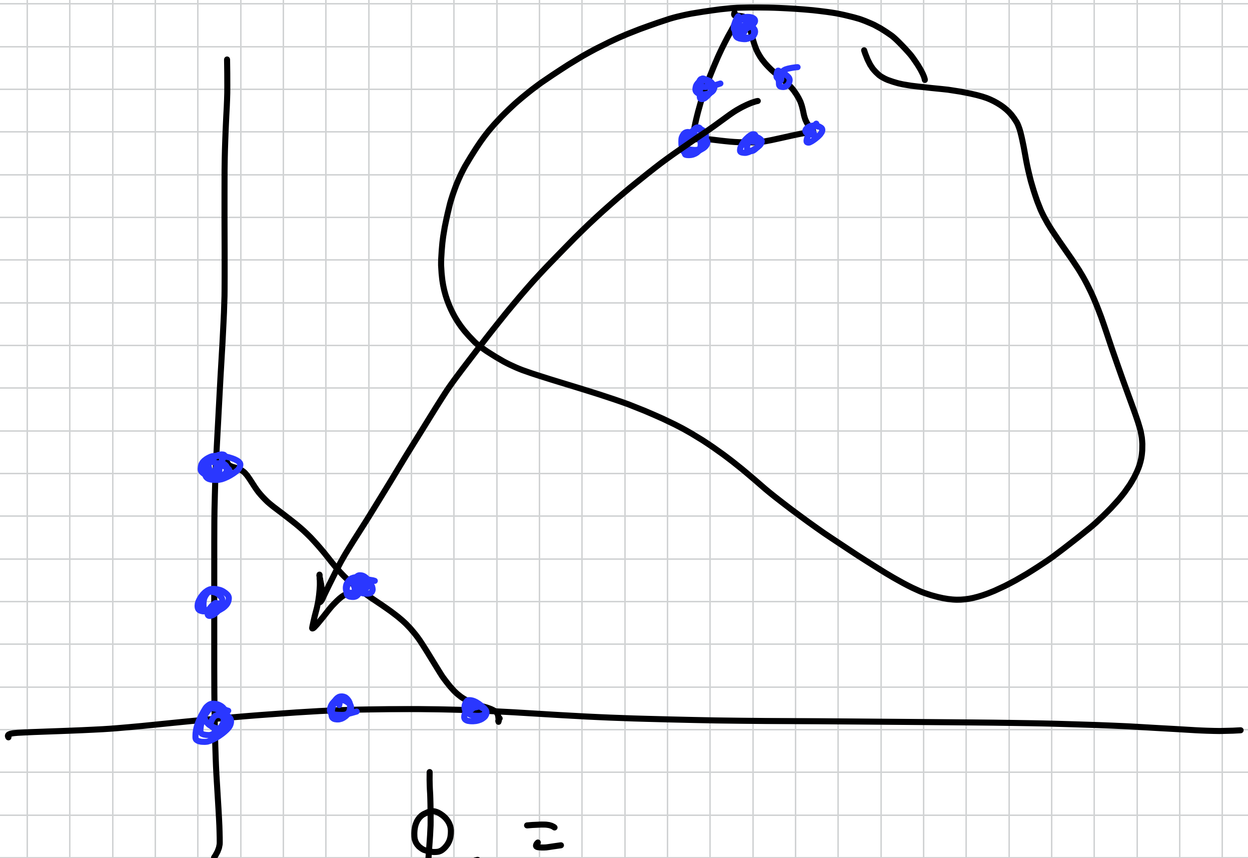






$$\phi_0 = 1 - x - y$$
$$\phi_1 = x$$
$$\phi_2 = y$$

$$\int f(x,y)\,\phi_i(x,y)\,dx\,dy$$



$\phi_0 =$

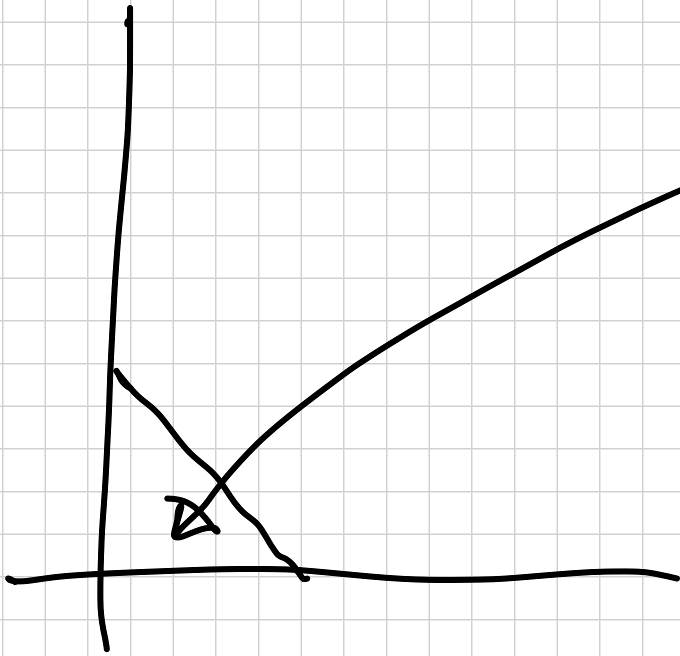
ϕ_1

ϕ_2

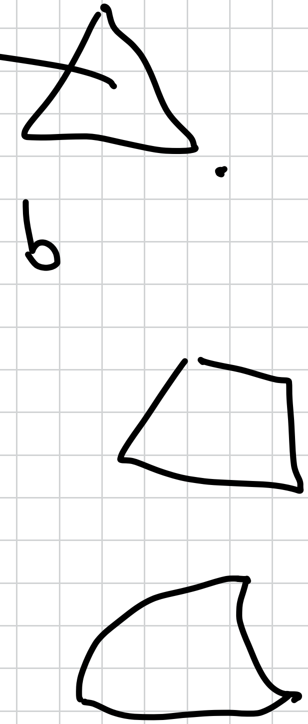
ϕ_3

ϕ_4

ϕ_5



$$F = A \cdot \begin{bmatrix} x \\ y \end{bmatrix} + b$$



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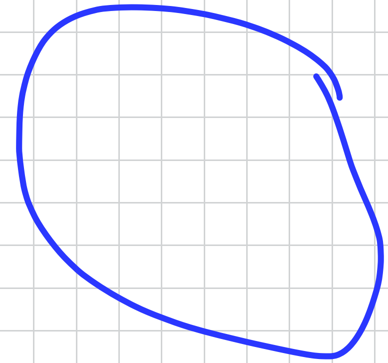

let Ω be an open domain

let $f \in H^1$

$$-\nabla \cdot \nabla u + u = f$$

$$u(x) = 0 \qquad x \in \partial\Omega$$

let $V = H^1_0(\Omega)$

$$\int_\Omega -\nabla\cdot\nabla u \, v + u v \, dx = \int f v \, dx$$

$$\cancel{-\int_{\partial\Omega} n \cdot \nabla u \, v} = 0$$

I.B.P. $\Rightarrow$

$$\int_\Omega \nabla u \cdot \nabla v + u v \, dx = \int_\Omega f v \, dx$$

Find $u \in H^1_0$ s.t.

$$\underbrace{\langle \nabla u, \nabla v \rangle + \langle u, v \rangle}_{\substack{a(u,v) \\ \text{bilinear form}}} = \langle f, v \rangle \quad \forall v$$

$$= \underbrace{g(v)}_{\text{linear functi.}}$$

Let $(V, \|\cdot\|_V)$ be a Banach Space.

A <u>linear</u> <u>functional</u> is a linear function

$$g : V \to \mathbb{R}$$

A linear functional is <u>bounded</u> if (or continuous)

$$|g(v)| \le c \cdot \|v\| \quad \forall\, v \in V.$$

Let V' = space of all <u>bounded</u> <u>linear</u> <u>functionals</u> on V.

dual space

with norm $\|g\|_{V'} = \sup_{v \in V} \frac{|g(v)|}{\|v\|_V}$

Back to the problem:

$$g(v) = \langle f, v \rangle$$

$\uparrow$ L^2 inner prod.

$$= \int f \cdot v \, dx$$

is $g(\cdot)$ a b.l.f.?

$$|g(v)| = |\langle f, v \rangle|$$

$$\leq \|f\|_2 \|v\|_2$$

$f \in H^1$

$v \in H^1$

Let $g(\cdot)$ be a bounded linear functional.

Then there exists a unique $u \in V$ such that

$$g(v) = (u, v)_V$$

Riesz Representation Theorem

$\langle \ , \ \rangle \leftarrow \mathbb{C}^2$ inner prod.

$(\ , \ )_V \leftarrow V$ inner prod.

Linear Algebra

let $V = \mathbb{R}^n$

let $g : V \to \mathbb{R}$

$\underline{v} \in V$,

$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + \cdots v_n \underline{e}_n$

$\to \quad g(v) = v_1 g(\underline{e}_1) + v_2 g(\underline{e}_2) + \cdots v_n g(\underline{e}_n)$

let $w_i = g(\underline{e}_i)$

$= \langle \underline{v}, \underline{w} \rangle$

$\underline{RRT}$ proof-ish

Given g. Find u s.t. $g(v) = \langle v, u \rangle_V$

Let $w \in N(g)^{\perp}$ ← all vectors orthogonal to all of $N(g)$

↑ nullspace

Let $\alpha = g(w)$

Pick any $v \in V$.

$$\Rightarrow g(v) = \frac{g(w)}{\alpha} \cdot g(v)$$

$$= g\left(\frac{g(v)}{\alpha} w\right)$$

$$\Rightarrow g\Big(\underbrace{v - \frac{g(v)}{\alpha} w}_{z}\Big) = 0$$

$z \in N(g)$

$g(z) = 0$

$$(z, w) = 0$$

$$\Rightarrow \left(v - \frac{g(v)}{\alpha} w,\ w\right) = 0$$

$$\Rightarrow \frac{g(v)}{\alpha}(w, w) = (v, w)$$

$$\Rightarrow g(v) = \left(v,\ \underbrace{\frac{\alpha}{(w,w)}\, w}_{u}\right)$$

$$= (v, u)$$

Back to

$$\underbrace{\langle \nabla u, \nabla v \rangle + \langle u, v \rangle}_{a(u,v)} = \langle \underbrace{f}_{g(v)}, v \rangle$$

any function in L^2

know: $a(u,v) = (u,v)_{H^1}$

Given a b.l.f. $g(\cdot)$

There exists a unique u s.t.

$$g(v) = (u,v)_{H^1} \qquad \text{R.R.T.}$$

$\longrightarrow$ exists unique solution.

Definition 8.23: $\mathcal{V}$-Ellipticity

Given a Hilbert Space, $\mathcal{V}$, consider a bilinear form

$$a(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}. \tag{8.66}$$

$a(\cdot,\cdot)$ is <u>coercive</u> if there exists a constant $c_0 > 0$ such that

$$c_0 \|u\|_{\mathcal{V}}^2 \leq a(u,u) \quad \textit{for all } u \in \mathcal{V}, \tag{8.67}$$

and $a(\cdot,\cdot)$ is <u>continuous</u> if there exists a constant $c_1 > 0$ such that

$$|a(u,v)| \leq c_1 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}} \quad \textit{for all } u,\, v \in \mathcal{V}. \tag{8.68}$$

If $a(\cdot,\cdot)$ is both coercive and continuous on $\mathcal{V}$, then $a(\cdot,\cdot)$ is said to be $\mathcal{V}$-elliptic.

Theorem 8.24: Lax-Milgram theorem (symmetric)

Let $\mathcal{V}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{V}}$. Assume that $a(\cdot,\cdot)$ is a symmetric bilinear form that is coercive and continuous on $\mathcal{V}$. In addition, assume that $g(\cdot)$ is a bounded linear functional on $\mathcal{V}$. Then, there exists a unique $u \in \mathcal{V}$ such that

$$a(u,v) = g(v) \quad \textit{for all } v \in \mathcal{V}. \tag{8.69}$$