Finite Element Methods: Recap

- Recall the FEM approach to the Poisson equation with homogeneous Dirichlet/Neumann conditions,

  Find $u \in X_0^N \subset \mathcal{H}_0^1$ *such that, for all* $v \in X_0^N$,

$$
\begin{aligned}
a(v,u) &= a(v,\tilde{u}) & (1)\\
&= \int_\Omega \nabla v \cdot \nabla \tilde{u}\, dV & (2)\\
&= -\int_\Omega v \nabla^2 \tilde{u}\, dV \;+\; \int_{\partial\Omega} v\, \nabla \tilde{u} \cdot \hat{\mathbf{n}}\, dS & (3)\\
&= \int_\Omega v\, f\, dV & (4)\\
&= (v,f), & (5)
\end{aligned}
$$

  which has built-in projective (best-fit) and SPD properties and which automatically incorporates the boundary conditions.

- Essentially, (1) forces u to be the closest element in X_0^N to $\tilde{u}$, including the Dirichlet/Neumann conditions.

FEM: Element-Based Implementation

- Implementation comes down to 3 parts:

  (1) Integration, e.g.,

$$(v,u) \;=\; \int_{\Omega} v\,u\,dV \;=\; \sum_{e=1}^{E} \int_{\Omega^e} v\,u\,dV \;=\; \sum_{e=1}^{E} \int_{\hat{\Omega}} v^e\,u^e\,\mathcal{J}^e\,d\mathbf{r}. \qquad (6)$$

  (2) Restricting v and u to $X^N \subset \mathcal{H}^1$ (inter-element continuity).

  (3) Restricting v and u to $X_0^N \subset \mathcal{H}_0^1$ (enforcing Dirichlet conditions).

Integration

- Fortunately, integration is relatively easy.

- Let's ignore the boundary conditions for the moment and just consider functions v and u in our *local, discontinuous finite element space,* $=: X_L^N$, having the form

$$v(\mathbf{x})|_{\Omega^e} = \sum_{i=1}^{n_v} l_i(\mathbf{r}) v_i^e \tag{7}$$

$$u(\mathbf{x})|_{\Omega^e} = \sum_{i=1}^{n_v} l_j(\mathbf{r}) u_j^e. \tag{8}$$

- We have, for all $v,\ u\ \in\ X^N$,

$$
\begin{aligned}
(v,u) &= \int_{\Omega} v\,u\,dV & (9)\\
&= \sum_{e=1}^{E} \int_{\Omega^e} v\,u\,dV & (10)\\
&= \sum_{e=1}^{E} \int_{\hat{\Omega}} \left(\sum_{i=1}^{n_v} v_i^e\, l_i(\mathbf{r}) \right) \left(\sum_{j=1}^{n_v} u_j^e\, l_j(\mathbf{r}) \right) \mathcal{J}^e(\mathbf{r}) d\mathbf{r} & (11)\\
&= \sum_{e=1}^{E} \sum_{i=1}^{n_v} \sum_{j=1}^{n_v} v_i^e \left(\int_{\hat{\Omega}} l_i(\mathbf{r})\, l_j(\mathbf{r})\, \mathcal{J}^e(\mathbf{r}) d\mathbf{r} \right) u_j^e & (12)\\
&= \sum_{e=1}^{E} \sum_{i=1}^{n_v} \sum_{j=1}^{n_v} v_i^e B_{ij}^e u_j^e & (13)\\
&= \sum_{e=1}^{E} (\underline{v}^e)^T B^e \underline{u}^e. & (14)\\
&= (\underline{v}_L)^T B_L \underline{u}_L. & (15)
\end{aligned}
$$

- Here, $B^e_{ij} := \int_{\hat{\Omega}} l_i(\mathbf{r})\, l_j(\mathbf{r})\, \mathcal{J}^e(\mathbf{r}) d\mathbf{r}$ is the *local* mass matrix, and $\underline{u}^e$ is the local vector of unknown basis coefficients.

- Recall that $\underline{u}_L = [\underline{u}^1\ \underline{u}^2\ \ldots\ \underline{u}^E]$ is the vector containing vectors of *local* basis coefficents,

$$\underline{u}^e \;=\; \left[u^e_1\, u^e_2\, \ldots\, u^e_{n_v}\right]^T. \qquad (16)$$

- Thus, our integral is the inner product with the block-diagonal (local) mass matrix, B_L:

$$(v,u) \;=\; \begin{pmatrix} \underline{v}^1 \\ \underline{v}^2 \\ \vdots \\ \underline{v}^E \end{pmatrix}^T \begin{pmatrix} B^1 & & & \\ & B^2 & & \\ & & \ddots & \\ & & & B^E \end{pmatrix} \begin{pmatrix} \underline{u}^1 \\ \underline{u}^2 \\ \vdots \\ \underline{u}^E \end{pmatrix} \;=\; \underline{v}_L^T\, B_L\, \underline{u}_L. \qquad (17)$$

- It is clear that, once $\underline{u}^1, \ldots, \underline{u}^e, \ldots, \underline{u}^E$ are known, one can compute $\underline{w}_L = B_L \underline{u}_L$ in a highly parallel fashion.

- It is also clear that computation of $\underline{v}_L^T \underline{w}_L$ involves a *contraction* (i.e., vector reduction), which is less parallel (but still has a lot of parallel work).

- Note that $\underline{v}_L$, $\underline{u}_L$, B_L reflect *all* dofs in X_L^N.

- $E \times n_v$ coefficients.

- Not continuous.

- No boundary conditions.

- To set up a projector into X_0^N, we introduce a pair of matrices, Q (for continuity) and R (for Dirichlet conditions).

Restricting v, u to $X^N \subset \mathcal{H}^1$

- We enforce C^0 continuity as follows,

$$\textit{If}\ \mathbf{x}_i^e \;=\; \mathbf{x}_{i'}^{e'},\ \textit{then}\ u_i^e \;=\; u_{i'}^{e'}. \tag{18}$$

- To implement (18), we introduce a *local-to-global map*,

$$i_g \;=\; t(e, i_v) \qquad \textit{(or } t(i_v, e)\textit{, depending on implementation)} \tag{19}$$

$$\mathbf{x}_{i_v}^e \;=\; \mathbf{x}_{i_g} \tag{20}$$

$$u_{i_v}^e \;=\; u_{i_g} \tag{21}$$

- Suppose $\{i_g\}$ is contiguous on $[1 : \bar{n}]$, with $\bar{n} := \max i_g$.

- Then, for any $\underline{\bar{v}}$, $\underline{\bar{u}} \in \mathbb{R}^{\bar{n}}$,

$$i_g \;=\; t(e, i_v) \tag{22}$$

$$u_j^e \;=\; u_{i_g} \tag{23}$$

$$v_j^e \;=\; v_{i_g}, \tag{24}$$

for $i_v = 1 : n_v$, $e = 1 : E$, will yield local representations of $v(\mathbf{x})$ and $u(\mathbf{x})$ in $X^N \subset \mathcal{H}^1$.

- As noted earlier, the map (24), which is just a copy operation, can be implemented with a Boolean matrix Q, known also as a global-to-local map and as a scatter (or gather) operation:

$$\underline{v}_L = Q\bar{\underline{v}} \tag{25}$$

$$\underline{u}_L = Q\bar{\underline{u}}, \tag{26}$$

where, for matlab/octave,

$$Q = \texttt{sparse(1:nv*E,reshape(t',nv*E,1),1);} \tag{27}$$

- Thus, $\forall\, v,\, u \in X^N \subset \mathcal{H}^1$,

$$\begin{aligned}
(v,u) &= \underline{v}_L^T B_L \underline{u}_L & (28)\\
&= (Q\bar{\underline{v}})^T B_L (Q\bar{\underline{u}}) & (29)\\
&= \bar{\underline{v}}^T \underbrace{Q^T B_L Q}_{\bar{B}}\, \bar{\underline{u}} & (30)\\
&= \bar{\underline{v}}^T \bar{B} \bar{\underline{u}}. & (31)
\end{aligned}$$

- $\bar{B}$ is the *assembled* mass matrix, which accounts for function continuity at the element interfaces.

- Unlike B_L, $\bar{B}$ is no longer block-diagonal.

Restricting v, u to $X_0^N \subset \mathcal{H}_0^1$

- We have not yet imposed Dirichlet conditions on $\partial\Omega_D$.

- Here, we require

$$u_{i_g},\, v_{i_g} \;=\; 0 \,\forall\, \mathbf{x}_{i_g} \;\in\; \partial\Omega_D. \tag{32}$$

- Suppose $\mathcal{I}_b = \{i_b\}$ is an index subset of $[1 : \bar{n}]$ that points to all $\mathbf{x}_{i_b} \in \partial\Omega_D$.

- Let $R^T = [\underline{\hat{e}}_j]$, $j \notin \mathcal{I}_b$, where $\underline{\hat{e}}_j$ is the jth col. of the $\bar{n} \times \bar{n}$ identity matrix.

- Example (see notes)

- *For any* $\underline{u} \in \mathbb{R}^n$, $\bar{\underline{u}} = R^T \underline{u}$ will correspond to an element in the FEM space that is in X_0^N.

- Specifically, the required *local* coefficients will be:

$$\underline{u}_L \;=\; Q R^T \underline{u}, \tag{33}$$

  and will correspond to

  (2) continuous functions

  (3) functions that vanish on $\partial\Omega_D$.

- So, the final step in setting up the matrix is, $\forall\, v\, u \in X_0^N$,

$$
\begin{aligned}
(v,u) &= (R^T \underline{v})^T Q^T B_L Q R^T \, \underline{u} &(34)\\
&= \underline{v}^T R Q^T B_L Q R^T \, \underline{u} &(35)\\
&= \underline{v}^T \, \underbrace{R \bar{B} R^T}_{B} \, \underline{u} &(36)\\
&= \underline{v}^T B \underline{u}. &(37)
\end{aligned}
$$

Summary

- Spaces

$$\underline{u}_L \;=\; \left(\begin{array}{c} \underline{u}^1 \\ \vdots \\ uu^E \end{array} \right) \;\Longrightarrow\; u \;\in X_L^N \;\; \text{(discontinuous, no BC restrictions)}$$

$$\underline{u}_L \;=\; Q\bar{\underline{u}} \;\Longrightarrow\; u \;\in X^N \;\; \text{(continuous, no BC restrictions)}$$

$$\underline{u}_L \;=\; QR^T\underline{u} \;\Longrightarrow\; u \;\in X^N \;\; \text{(continuous, 0 on Dirichlet bdry)}$$

- Matrices

$$B_L \;\; - \;\; \text{block-diagonal} \tag{41}$$

$$\bar{B} = Q^T B_L Q \;\; - \;\; \text{assembled, with boundary points} \tag{42}$$

$$B = R\bar{B}R^T \;\; - \;\; \text{restricted, no Dirichlet points} \tag{43}$$