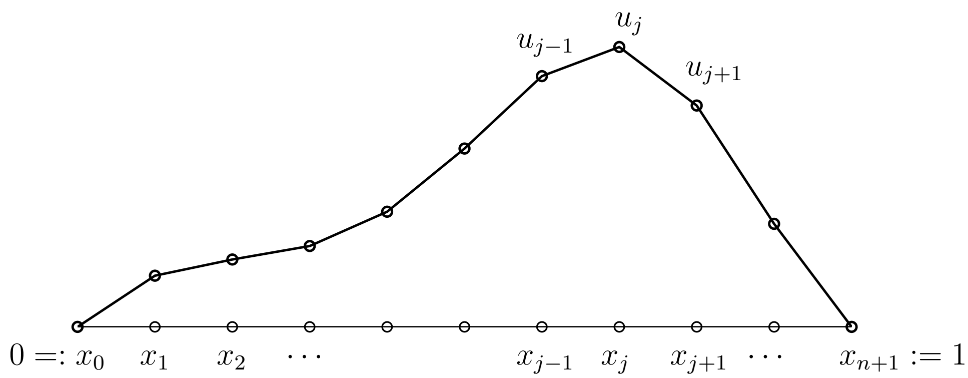
CS556 Iterative Methods Fall 2024 Homework 1.

Due Tuesday, Sep. 10, 5 PM.

1. The finite difference grid shown in the figure below uses a *uniform grid spacing*, Δx.



The governing system for the solution is $A\underline{u} = \underline{b}$ with A the $n \times n$ SPD matrix,

$$A = \frac{1}{\Delta x^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}, \tag{1}$$

and $\Delta x = 1/(n+1)$. In the case of uniform spacing, the eigenvalues for A, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, are

$$\lambda(A) = \frac{2}{\Delta x^2}(1 - \cos(k\pi\Delta x)) \in \left(\pi^2(1 + O(\Delta x^2)), 4(n+1)^2\right) \tag{2}$$

$$\in \left(\pi^2(1 + O(\Delta x^2)), \frac{4}{\Delta x^2}\right). \tag{3}$$

The lower bound of $\approx \pi^2$ is readily derived from the smallest eigenvalue of the continuous eigenproblem,

$$-\frac{d^2\tilde{u}}{dx^2} = \tilde{\lambda}\tilde{u}, \quad \tilde{u}(0) = \tilde{u}(1) = 0, \tag{4}$$

which has a minimum eigenvalue of π^2 associated with the eigenmode $\tilde{s}_k := \sin(k\pi x)$ for $k = 1$. *Every* reasonable discretization of (**??**) will accurately approximate the lower bound eigenpair $(\tilde{\lambda}_1, \tilde{s}_1)$ as n is increased. The largest eigenvalue is more difficult to estimate but can readily be found by power iteration.

An inexpensive way to find a lower bound of $\lambda_n = \lambda_{\max}$ is to use the relationship,

$$\lambda_n = \max_{\underline{z}} \frac{\underline{z}^T A \underline{z}}{\underline{z}^T \underline{z}} = \max_{\|\underline{y}\|=1} \underline{y}^T A \underline{y}. \tag{5}$$

(a) Consider an arbitrary unit vector $\underline{y} = [0\,0\ldots0\,1\,0\ldots0]^T$ and show, without using (2) (i.e., just from the definition of A), that $\lambda_{\max}(A) = O(n^2)$ and that the condition number of A, $\kappa(A) = \lambda_{\max}/\lambda_{\min}$ is $O(n^2)$.

(b) Suppose now that you have a *nonuniform* grid spacing with $\Delta x_j := x_j - x_{j-1}$, for $j = 1, \ldots, n+1$. A finite difference formula about the point x_j is derived as follows.

$$-\left.\frac{d^2u}{dx^2}\right|_{x_j} \approx -\frac{1}{\overline{\Delta x}}\left[\frac{u_{j+1}-u_j}{\Delta x_{j+1}} - \frac{u_j - u_{j-1}}{\Delta x_j}\right], \tag{6}$$

where $\overline{\Delta x} = (\Delta x_j + \Delta x_{j+1})/2$. Assuming that the grid spacing is slowly varying (i.e., $\Delta x_j = C_j \Delta x_{j+1}$, where, say, $\frac{1}{2} < C_j < 2$), show that $\kappa(A) = O(1/\Delta x^2_{\min})$ for the case of variable grid spacing.

(c) Let D =diag(a_{ii}) be the diagonal of A.* Use the Gershgorin circle theorem to show that the eigenvalues of $D^{-1}A$ are on the interval [0,2] for both the uniform and variable grid spacing cases.

(d) Consider the case where x_j is given by a Chebyshev distribution,

$$x_j = \frac{1}{2}(1 + cos\theta_j) \tag{7}$$

$$\theta_j = \pi\left(\frac{j}{n+1} - 1\right), \quad j = 0, \ldots, n+1. \tag{8}$$

- How does the maximum grid spacing vary with n?
- How does the minimum grid spacing vary with n?
- Does the "slowly varying" condition hold? (Check the ratio for the minimum grid spacing.)
- Use octave/matlab to find the min and max eigenvalues.
- Do your estimates from above hold? (Demonstrate by plotting $\lambda_{\min}$ and $\lambda_{\max}$ vs. n for, say, n = 1, 2, 4, 8,. . . ,128, on a log-log scale.

*In matlab, `d=diag(A)` returns the *vector* containing the diagonal elements of A, and `D=diag(d)` converts this vector to a (full!) diagonal matrix.

2. Consider powers of A for the tridiagonal matrices of the preceding question, $M_k := A^k$, for $0 \leq k \ll n$. Give a succinct formula for the number of nonzeros in M_k. You may (and should) neglect end effects—we just need an estimate for $n \gg k$.

3. Consider application of A from the preceding problem to a unit vector, $\underline{e}_j$, which is the jth column of the identity matrix. Let $\underline{w}_0 = \underline{e}_j$ and $\underline{w}_k = A\underline{w}_{k-1}$. Out of all possible j, what is the *minimum* number of iterations (i.e., value of k) that is required to ensure that $\underline{w}_k$ has no zero entries? (For example, for $k = 0$, $\underline{w}_k$ has only one nonzero entry. For $k = 1$ it has more than one. How many?, etc., You can ignore the possibility of cancellation.)

4. Using the notation of the preceding questions, if we set $\underline{b} = \underline{e}_j$, show that the solution of $A\underline{u} = \underline{b}$ corresponds to the jth column of A^{-1}. For A with uniform spacing and $n = 20$, plot on a single graph the columns of A^{-1} vs. x_i (i.e., 20 plots on a single graph).

   - What do your plots tell you about the sign of the entries of A^{-1}?
   - What do your plots tell you about the number of nonzeros in A^{-1}?
   - What do your plots tell you about the communication implied by the solution of $A\underline{u} = \underline{b}$ when $\underline{u}$ and $\underline{b}$ are distributed vectors across $P \gg 1$ processors?