Cost Analysis: Jacobi Iteration

- Recall our Jacobi iteration from Lec. 1, with $D := \text{diag}(A)$.
- We can express this in two ways,

Analysis:

$$\begin{aligned}&\underline{x}_0 = 0\\&\textit{for } k = 1 : k_{\max}\\&\quad \underline{x}_k \;=\; \underline{x}_{k-1} \;+\; D^{-1}(\underline{b} - A\underline{x}_{k-1})\end{aligned}$$

Practice:

$$\begin{aligned}&\underline{x} = 0\\&\textit{for } k = 1 : k_{\max}\\&\quad \underline{x} \;=\; \underline{x} \;+\; D^{-1}(\underline{b} - A\underline{x})\end{aligned}$$

- Two main questions arise re. complexity:

  (1) How much work, w, per iteration?

  (2) How many iterations, k?

- The total work is $W \;=\; w\,k$.

- To address the first, consider more realistic loops:

$$
\begin{aligned}
&\underline{x} = 0, \;\; \underline{r} = \underline{b} \\
&\textit{for } k = 1 : k_{\max} \\
&\quad \rho \;=\; \|\underline{r}\|_2 \;=\; \sqrt{\underline{r}^T\underline{r}} \\
&\quad \textit{if } \rho < \textit{ tol, break.} \\
&\quad \underline{x} \;=\; \underline{x} \;+\; D^{-1}\underline{r} \\
&\quad \underline{r} \;=\; \underline{b} \;-\; A\underline{x} \\
&\textit{end}
\end{aligned}
$$

- A better approach is:

$$
\begin{aligned}
&\underline{x} = 0, \;\; \underline{r} = \underline{b}, \;\; \rho_0 \;=\; \|\underline{r}\|_2 \;=\; \sqrt{\underline{r}^T\underline{r}} \\
&\textit{for } k = 1 : k_{\max} \\
&\quad \rho \;=\; \|\underline{r}\|_2 \;=\; \sqrt{\underline{r}^T\underline{r}} \\
&\quad \textit{if } \rho/\rho_0 \;<\; \textit{tol, break.} \\
&\quad \underline{s} \;=\; D^{-1}\underline{r} \\
&\quad \underline{x} \;=\; \underline{x} \;+\; \underline{s} \\
&\quad \underline{r} \;=\; \underline{r} \;-\; A\underline{s} \\
&\textit{end}
\end{aligned}
$$

- **Q:** *How many operations for each step, as a function of d?*

- Quick summary:

$$
\begin{aligned}
d = 1: &\quad 9n \cdot k \\
d = 2: &\quad 13n \cdot k \\
d = 3: &\quad 17n \cdot k
\end{aligned}
$$

- We see that the cost *per iteration* is only weakly dependent on d!

- What about the *number* of iterations?

- To analyze this question, we'll need some norms to measure the error.

- Let's start with the vector 2-norm,

$$\|\underline{x}\|_2 \;:=\; \sqrt{\underline{x}^T \underline{x}} = \left(\sum_j^n x_j^2 \right)^{\frac{1}{2}} . \tag{1}$$

- With this vector norm, we have an associated *matrix norm*,

$$\|A\|_2 \;:=\; \max_{\underline{x} \in \mathbb{R}^n} \frac{\|A\underline{x}\|_2}{\|\underline{x}\|_2} \tag{2}$$

$$\;:=\; \max_{\|\underline{x}\|=1} \|A\underline{x}\|_2. \tag{3}$$

- We see that $\|A\|_2$ is identified with the *maximum stretching* (growth) of any input vector $\underline{x}$.

- For the case of $A = A^T$, we have $\|A\|_2 \;=\; \rho(A)$.

- Therefore, we know the 2-norm of A for our finite difference matrices!

- Which matrix do we need the 2-norm for to understand the error behavior of Jacobi iteration?