CS556 Iterative Methods Fall 2020
Eigenvalues from PCG

Consider the following PCG algorithm with A and preconditioner M SPD.

$$\text{for } k = 1 : n \tag{1}$$

$$\text{Solve } M\underline{z}_k = \underline{r}_{k-1},\ \rho_k = \underline{z}_k^T \underline{r}_{k-1},\ \beta_k = \tfrac{\rho_k}{\rho_{k-1}},\ \text{if } k\text{=1},\ \beta_k = 0 \tag{2}$$

$$\underline{p}_k \;=\; \underline{z}_k \;+\; \beta_k \underline{p}_{k-1} \tag{3}$$

$$\underline{w}_k \;=\; A\underline{p}_k,\ \ \gamma_k = \underline{p}_k^T \underline{w}_k,\ \ \alpha_k = \frac{\rho_k}{\gamma_k} \tag{4}$$

$$\underline{x}_k \;=\; \underline{x}_{k-1} \;+\; \alpha_k \underline{p}_k \tag{5}$$

$$\underline{r}_k \;=\; \underline{r}_{k-1} \;-\; \alpha_k \underline{w}_k \tag{6}$$

$$\text{end} \tag{7}$$

Let

$$R \;=\; [\underline{r}_0\ \underline{r}_1\ \cdots\ \underline{r}_{k-1}], \tag{8}$$

$$P \;=\; [\underline{p}_1\ \underline{p}_2\ \cdots\ \underline{p}_k], \tag{9}$$

$$Z \;=\; [\underline{z}_1\ \underline{z}_2\ \cdots\ \underline{z}_k], \tag{10}$$

and

$$B \;=\; \begin{bmatrix} 1 & -\beta_2 & & & \\ & 1 & -\beta_3 & & \\ & & 1 & -\beta_4 & \\ & & & \ddots & \end{bmatrix}. \tag{11}$$

Note that P^TAP =diag(γ_i). We also have that $Z = PB$, so

$$Z^TAZ \;=\; \tilde{T}, \tag{12}$$

where $\tilde{T}$ is a $k \times k$ tridiagonal matrix. Moreover, $Z^TMZ = Z^TR$ =diag(ρ_i) $=: \Delta^2$. From these, we can find approximate eigenvectors and eigenvalues for the generalized eigenvalue problem, $A\underline{s}_j = \lambda_j M\underline{s}_j$. Let's consider λ_n, the eigenvalue that maximizes the Rayleigh quotient,

$$\lambda_n \;=\; \max_{\underline{s}\in\mathbb{R}^n} \frac{\underline{s}^TA\underline{s}}{\underline{s}^TM\underline{s}}. \tag{13}$$

The approximated value is

$$\lambda_n \;\approx\; \max_{\underline{z}\in Z} \frac{\underline{z}^TA\underline{z}}{\underline{z}^TM\underline{z}} \;=\; \max_{\underline{y}\in\mathbb{R}^k} \frac{\underline{y}^TZ^TAZ\underline{y}}{\underline{y}^TZ^TMZ\underline{y}} \;=\; \max_{\underline{y}\in\mathbb{R}^k} \frac{\underline{y}^T\tilde{T}\underline{y}}{\underline{y}^T\Delta^2\underline{y}} \;=\; \mu_k, \tag{14}$$

where μ_k is the maximum eigenvalue for the $k \times k$ generalized eigenvalue problem,

$$\tilde{T}\underline{y}_j \;=\; \mu_j \Delta^2 \underline{y}_j \tag{15}$$

Here, Δ^2 is a diagonal matrix. Define $\underline{u} = \Delta\underline{y} \longrightarrow \underline{y} = \Delta^{-1}\underline{u}$. The right-most Rayleigh quotient in the preceding equation becomes

$$\mu_k \;=\; = \max_{\underline{u}\in\mathbb{R}^k} \frac{\underline{u}^TT\underline{u}}{\underline{u}^T\underline{u}}, \tag{16}$$

where $T := \Delta^{-1}\tilde{T}\Delta^{-1}$.