## A First Preconditioner

- Consider the 1D model problem $A\underline{x} = \underline{b}$, with

$$A = \frac{1}{h^2}\,\text{tridiag}[-1 \;\; 2 \;\; -1], \tag{1}$$

  and let $D = \text{diag}(A) = \frac{2}{h^2} I$.

- From Gershgorin, the eigenvalues of $D^{-1}A$ satisfy $0 < \lambda_k \leq 2$, where the strict inequality on the left derives from the fact that $A$ is SPD.

- We also know that the extreme eigenvalues of $D^{-1}A$ satisfy

$$\lambda_n(A) \sim 2 \tag{2}$$

$$\lambda_1(A) \sim \frac{h^2}{2}\pi^2 \longrightarrow 0 \text{ as } h \longrightarrow 0 \tag{3}$$

  and the condition number is $\kappa(D^{-1}A) \sim \frac{\pi^2 n^2}{4}$.

- For this scaled system, it is the small eigenvalues that are the source of difficulty.

- Imagine if we could modify just the upper half of the spectrum such that all eigenvalues were $\leq 1$.

- Then we would improve the condition number by only a factor of 2, to $\kappa \sim \frac{\pi^2 n^2}{2}$.

- On the other hand, if we could shift the lower end of the spectrum such that $1 < \lambda_k < 2$, the condition number would $\leq 2$, which is a *huge* improvement.

- This is the goal of coarse-grid solvers.

- To begin, we remark that application of diagonal preconditioner does not change the condition number in the case when the diagoanl of $A$ is constant.

- It does help, however, if the diagonal elements of $A$ span a broad range of values, e.g., if the mesh spacing is highly nonuniform.

- In the present case, it will be useful because it scales the maximum eigenvalues so that they satisfy $\lambda_k(D^{-1}A) \in [0, 2]$.

- Let the eigenvalues of $D^{-1}A$ be ordered as $\lambda_1 < \lambda_2 \leq \cdots \lambda_n$, and denote the set of "small" eigenvalues as those $\lambda_k$ for which $k \leq c$, where $c \ll n$ is a wavenumber cut-off value.

- Consider now a "coarse-space" preconditioner, defined as

$$M_c^{-1} \;=\; J_c A_c^{-1} J_c^T, \tag{4}$$

with $A_c := J_c^T A J_c$, and $J_c \in \mathbb{R}^{n\times c}$ is a low-dimensional space with only $c$ columns.

- Recall that, in our PCG iteration, $\underline{r}_k \equiv A\underline{e}_k$, *always.*

- If we use $M_c$ as a preconditioner, our tentative search direction is

$$\underline{z}_c \;=\; M_c^{-1}\underline{r} \;=\; M_c^{-1}A\underline{e} \tag{5}$$

$$=\; J_c A_c^{-1} J_c^T A\underline{e} \tag{6}$$

$$=\; J_c (J_c^T A J_c)^{-1} J_c^T A\underline{e} \tag{7}$$

$$=\; \Pi_A(J_c)\,\underline{e}, \tag{8}$$

which is the $A$-orthogonal projection of the current error onto $\mathcal{R}(J_c)$.

- In other words, using $\underline{z}_c = M_c^{-1}\underline{r}$ produces the *closest element* in the column space of $J_c$ to the desired error correction, $\underline{e}$.

- Notice also that it is relatively inexpensive to compute $\underline{z}_c$ because we first compute $J_c^T\underline{r}$ in $2nc$ operations, then apply $A_c^{-1}$, which requires $2c^2$ operations, then assemble the linear combination of the columns of $J_c$, which requires $2nc$ operations.

- If the columns of $J_h$ are sparse, the work complexity drops to $O(n)$.

- Unfortunately, the low-cost preconditioner $M_c^{-1}$ suffers from being rank-deficient.

- It is incapable of generating in components of the solution in $\mathcal{R}^{\perp}(J_c)$, the orthogonal complement of $\mathcal{R}(J_c)$.

- To remedy this situation, we consider an equally-inexpensive precondtioning step,

$$\underline{z} \;=\; M_c^{-1}\,\underline{r} \;+\; D^{-1}\underline{r}. \tag{9}$$

- Now, the preconditioner, $M^{-1} = M_c^{-1} + D^{-1}$, is of full rank.

- To see how this is beneficial, let's assume that diag($A$) is a constant and that $J_h$ spans the first $c$ eigenvectors of $A$, $J_h = [\underline{s}_1\ \underline{s}_2\ \dots\ \underline{s}_c]$, where $\underline{s}_i^T\underline{s}_j = \delta_{ij}$.

- Thus, $M^{-1}$ and $A$ share the same eigenvectors.

- Suppose $\underline{e} = \sum_k \hat{e}_k \underline{s}_k$.

- We have,

$$
\begin{aligned}
A_c &= J_c^T A J_c \;=\; [\underline{s}_1 \ldots \underline{s}_c]^T A [\underline{s}_1 \ldots \underline{s}_c] && (10)\\
&= \mathrm{diag}(\lambda_k), \quad k = 1, \ldots, c, && (11)\\
A_c^{-1} &= \mathrm{diag}(\lambda_k^{-1}), \quad k = 1, \ldots, c, && (12)\\
J_c^T \underline{r} &= J_c^T A (\sum_{k=1}^{n} \hat{e}_k \underline{s}_k) && (13)\\
&= \begin{bmatrix} \lambda_1 \hat{e}_1 \\ \lambda_2 \hat{e}_2 \\ \vdots \\ \lambda_c \hat{e}_c \end{bmatrix} && (14)\\
A_c^{-1} J_c^T \underline{r} &= \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \\ \hat{e}_c \end{bmatrix} && (15)\\
& && (16)\\
J A_c^{-1} J_c^T \underline{r} &= \sum_{k=1}^{c} \hat{e}_k \underline{s}_k. && (17)
\end{aligned}
$$

- Clearly, if $\underline{e} \in \mathcal{R}(J_c)$, we have $\underline{z} = \underline{e}$.

- If $\underline{e} \in \mathcal{R}^{\perp}(J_c)$, we have $\underline{z} = 0$.

- Thus, the eigenvalues of $M_c^{-1}A$ are either 0 or 1, depending on whether $k > c$ or $k \leq c$.

- Recall that the eigenvalues of $D^{-1}A$ are in [0,2].

- So, the low-end of the spectrum of $M^1 A$ is shifted by +1 through the presence of $M_c^{-1}$, as illustrated below.

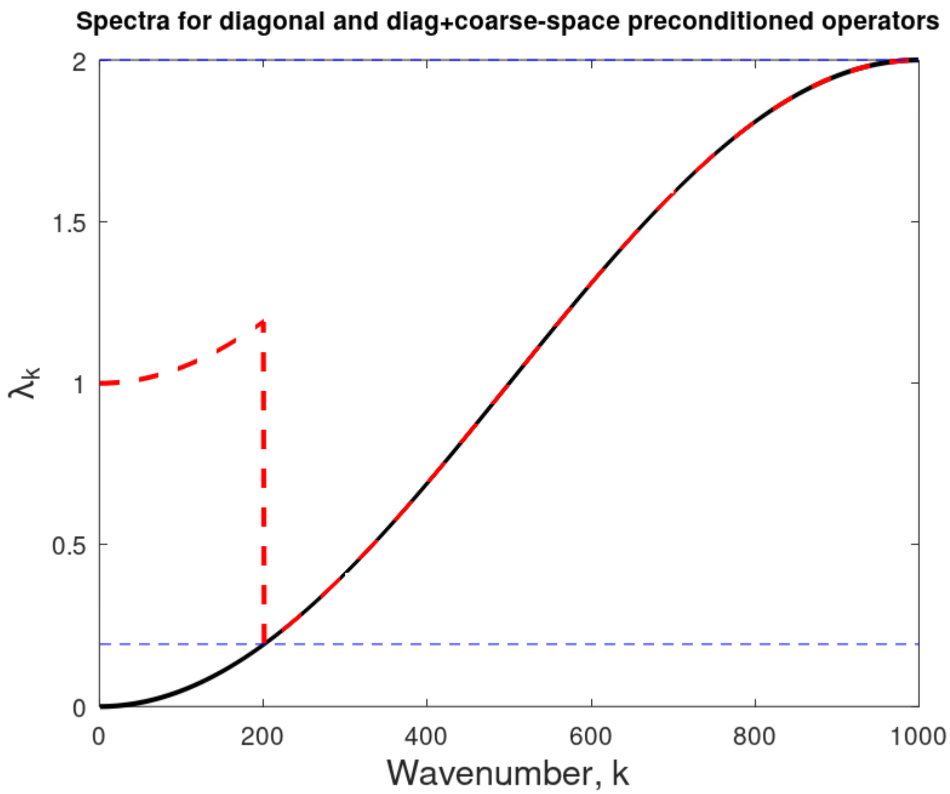


Figure 1: Spectrum for $D^{-1}A$ (black) and $M^{-1}A$ (red-dash) for 1D Poisson problem.

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- Notice that, crucially, the condition number of $M^{-1}$ is controlled by $\lambda_n/\lambda_{c+1}$, and not by the details of the shifted part of the spectrum.

- This observation allows for many simplications, including
  - We can replace the columns of $J_c$ with a coarse-grid interpolant that interpolates from a few ($c$) nodes to the original $n$ points.
  - Using piecewise linear interpolation yields a sparse matrix $J_h$.
  - We can replace $A_c = J_c^T A J_c$ with an approximation on a coarse mesh with grid spacing $H = L/(c+1)$. (Note, we may need to rescale the columns of $J_c$ in this case, but the piecewise linear interpolants should be ok because the columns of $J_c$ have nearly unit-norm.)