Last time:
- GPA uniqueness
- Calderón identities

Next:
- Green's formula
- Jump relations

TODAY

law 3 graded

hus - 3 be - 4
## Boundary Value Problems: Overview

<table>
<thead>
<tr>
<th></th>
<th>Dirichlet</th>
<th>Neumann</th>
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<tbody>
<tr>
<td><strong>Int.</strong></td>
<td>( \lim_{x \to \partial \Omega^-} u(x) = g )</td>
<td>( \lim_{x \to \partial \Omega^-} \hat{n} \cdot \nabla u(x) = g )</td>
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<td></td>
<td>unique</td>
<td>may differ by constant</td>
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<tr>
<td><strong>Ext.</strong></td>
<td>( \lim_{x \to \partial \Omega^+} u(x) = g )</td>
<td>( \lim_{x \to \partial \Omega^+} \hat{n} \cdot \nabla u(x) = g )</td>
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<tr>
<td></td>
<td>( u(x) = \begin{cases} O(1) &amp; 2D \text{ as }</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td>unique</td>
<td>unique</td>
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</tbody>
</table>

with \( g \in C(\partial \Omega) \).

What does \( f(x) = O(1) \) mean? (and \( f(x) = o(1) \)?)

Dirichlet uniqueness: why?

Neumann uniqueness: why?

Truth in advertising: Missing assumptions on \( \Omega \)?
**What’s a DtN map?**

**Next mission:** Find IE representations for each.

|               | Dirichlet                | Neumann
|---------------|--------------------------|----------
| **in L**      |                          |          |
|              | $u(x) = D\sigma(x)$      | $u(x) = S\sigma(x)$ |
|              | $IE \left( \frac{1}{Z} - D \right)\sigma(x) = 0$ | $IE: \left( \frac{1}{Z} + S' \right)\sigma(x)$ |
| **ext L**     |                          |          |
|              | $u(x) = D\sigma(x)$      | $u(x) = S\sigma(x)$ |
|              | $IE \left( \frac{1}{Z} + D \right)\sigma(x) = 0$ | $IE: \left( \frac{1}{Z} - S' \right)$ |

Note: **not compact; boundary**
Uniqueness of Integral Equation Solutions

Theorem 17 (Nullspaces [Kress LIE 2nd ed. Thm 6.20])

- \( N(I/2 - D) = \{0\} \)
- \( N(I/2 + D) = \text{span}\{1\}, \ N(I/2 + S') = \text{span}\{\psi\}, \)
  where \( \int \psi \neq 0. \)

Show \( N(I/2 - D) = \{0\}. \)

Show \( N(I/2 - S') = \{0\}. \) (FH)

Show \( N(I/2 + D) \supset \text{span}\{1\}. \)

What extra conditions on the RHS do we obtain?
Show $N\left( \frac{\pi}{2} - D \right) = \{0\}$.

Suppose: $\frac{\pi}{2} - D \psi = 0$. To show: $\psi = 0$

$u(x) = D\psi(x)$. $\Delta u(x) = 0$ off of $\Gamma$. 

$$\lim_{x \to \Gamma} u^- = D\psi - \psi/2 = 0$$

$u/\text{int} < 0$ because of int BVP uniqueness.

$(\partial_n u)^- = 0 = (\partial_n u)^+$

$\Rightarrow u|_{\text{ext}} = 0$ By jump cond. $\psi = u_+ - u_- = 0$
→ “Clean” Existence for 3 out of 4.
Patching up Exterior Dirichlet (skipped)

Problem: $N(I/2 + S') = \{\psi\}$...but we do not know $\psi$.

Use a different kernel:

$$\hat{n} \cdot \nabla_y G(x, y) \rightarrow \int (\hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}}) \sigma(y) \, dy$$

Note: Singularity only at origin! (assumed $\in \Omega$)

- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce $u = 0$ on exterior.
- $|x|^{n-2}u(x) = ?$ on exterior
- Thus $\int \phi = 0$. Contribution of the second term?
- $\phi/2 + D\phi = 0$, i.e. $\phi \in N(I/2 + D) =$?
\[ \Delta u = 0 \quad \Rightarrow \quad \Delta u + \kappa^2 u = 0 \]

\[ \Delta u - \kappa^2 u = 0 \]

\[ \n \approx e^{i \omega t} \tilde{u} \]

\[ \nabla^2 u = \Delta u \]

\[ (iu)^2 \]

- \[ \Delta u \] has nonnegative eigenvalues

\[ \Delta u \]

\[ (\text{neg. definite}) \quad \Delta u - \kappa^2 \]

\[ \Delta u + \kappa^2 \]

"back" Helmholtz

"good" Yukawa Poisson-Boltzmann
\[ \Delta u + \lambda u^2 = 0 \]
\[ u = 0 \]
• Existence/uniqueness?

→ Existence for 4 out of 4.

Remaining key shortcoming of IE theory for BVPs?
Domains with Corners

What’s the problem? \textit{(Hint: Jump condition for constant density)}

At corner $x_0$: (2D)

\[
\lim_{x \to x_0^\pm} = \int_{\partial \Omega} \hat{n} \cdot \nabla_y G(x, y) \phi(y) dy \pm \frac{1}{2} \left( \text{opening angle on } \pm \text{ side} \right) \phi 
\]

\[\rightarrow\] non-continuous behavior of potential on $\Gamma$ at $x_0$

What space have we been living in?

Fixes:

- $I$ + Bounded (Neumann) + Compact (Fredholm)
• Use $L^2$ theory
  (point behavior “invisible”)
Numerically: Needs consideration, but ultimately easy to fix.
8.2 Helmholtz
Where does Helmholtz come from?

Derive the Helmholtz equation from the wave equation

$$\partial_t^2 U = c^2 \Delta U,$$
The prototypical Helmholtz BVP: A Scattering Problem

Ansatz:

\[ u^{\text{tot}} = u + u^{\text{inc}} \]

Solve for scattered field \( u \).