TODAY:

- project pres. signup
- hw 3, 4, grading
- hw 5: coming
\[ \|A\|_\infty \geq \| \sum \phi(x,y) \rho(y) \phi(y) \| \]

I: compact \( \rightarrow \) \( C(\Gamma) \rightarrow C(\Gamma) \)

\((I - \text{compact}) \phi = 0 \)

\[
\lim_{x \to 0^+} D\phi - \lim_{x \to 0^-} D\phi = \phi
\]
\[ \dim(N(I-A)) < \infty \]

\[ \forall \in N(I-A) \]

\[ A\gamma = \gamma \]

\[ 0 \in \sigma(A) \]

Suppose \( 0 \in \sigma(A) \).

\[ \Rightarrow A \text{ is injective} \]

\[ \Rightarrow A^* \text{ is surjective} \]

\[ \Rightarrow A^{-1} \text{ exists} \]

\[ \Rightarrow I = AA^{-1} \text{ compact} \]
\[ u(x) = \int_{\mathbb{R}} \delta(x) \, dx \]
Recap: Layer potentials

\[
\begin{align*}
S\sigma(x) & := \int_{\Gamma} G(x - y)\sigma(y)ds_y \\
L'\sigma(x) & := \text{PV} \hat{n} \cdot \nabla_x \int_{\Gamma} G(x - y)\sigma(y)ds_y \\
L\sigma(x) & := \text{PV} \hat{n} \cdot \nabla_y G(x - y)\sigma(y)ds_y \\
\Gamma\sigma(x) & := \text{f.p.} \hat{n} \cdot \nabla_y G(x - y)\sigma(y)ds_y
\end{align*}
\]

Important for us: Recover ‘average’ of interior and exterior limit without having to refer to off-surface values.

\[
(A\gamma, \psi) = (\gamma, A^*\psi)
\]

\[
\int \hat{A}\gamma(x) \psi(x) dx = \int \psi(x) A^*\gamma(x) dx
\]
\[ \int \int g(x,y) \phi(x) \psi(x) \, dx \]
Green's Theorem

**Theorem 11 (Green's Theorem [Kress LIE 2nd ed. Thm 6.3])**

\[
\begin{align*}
\int_{\Omega} u \Delta v + \nabla u \cdot \nabla v &= \int_{\partial \Omega} u(\hat{n} \cdot \nabla v) \, ds \\
\int_{\Omega} u \Delta v - v \Delta u &= \int_{\partial \Omega} u(\hat{n} \cdot \nabla v) - v(\hat{n} \cdot \nabla u) \, ds
\end{align*}
\]

If \( \Delta v = 0 \), then

\[
\int_{\partial \Omega} \hat{n} \cdot \nabla v = 0
\]

What if \( \Delta v = 0 \) and \( u = G(|y - x|) \) in Green's second identity?

**Green's Formula**

\[
0 = \sum_{\partial \Omega} (\hat{n} \cdot \nabla v) \, ds
\]
\[- \int v(y) \delta(x-y) \, d\lambda = \int \varphi \big( \delta_n \psi \big) - D(v) \]
Theorem 12 (Green’s Formula [Kress LIE 2nd ed. Thm 6.5]) If $\Delta u = 0$, then

$$(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} 
  u(x) & x \in \Omega \\
  \frac{u(x)}{2} & x \in \partial \Omega \\
  0 & x \notin \Omega
\end{cases}$$

Suppose I know ‘Cauchy data’ $(u|_{\partial \Omega}, \hat{n} \cdot \nabla u|_{\partial \Omega})$ of $u$. What can I do?

What if $\Omega$ is an exterior domain?

What if $u = 1$? Do you see any practical uses of this?
Things harmonic functions (don’t) do

**Theorem 13 (Mean Value Theorem [Kress LIE 2nd ed. Thm 6.7])** If $\Delta u = 0$, 

$$u(x) = \int_{B(x,r)} u(y) dy = \int_{\partial B(x,r)} u(y) dy$$

Define $\bar{f}$?

Trace back to Green’s Formula (say, in 2D):

**Theorem 14 (Maximum Principle [Kress LIE 2nd ed. 6.9])** If $\Delta u = 0$ on compact set $\bar{\Omega}$: 

$u$ attains its maximum on the boundary.
Suppose it were to attain its maximum somewhere inside an open set...

What do our constructed harmonic functions (i.e. layer potentials) do there?
Jump relations

Let $[X] = X_+ - X_-$. (Normal points towards “+” = “exterior”.)

[Kress LIE 2nd ed. Thm. 6.14, 6.17, 6.18]
\[ \lim_{x \to x_0 \pm} (S' \sigma) = \left( S' \mp \frac{1}{2} l \right) (\sigma)(x_0) \Rightarrow [S' \sigma] = -\sigma \]

\[ \lim_{x \to x_0 \pm} (D \sigma) = \left( D \mp \frac{1}{2} l \right) (\sigma)(x_0) \Rightarrow [D \sigma] = \sigma \]

Truth in advertising: Assumptions on $\Gamma$?

Sketch the proof for the single layer.

Sketch proof for the double layer.
Green’s Formula at Infinity  

$\Omega \subseteq \mathbb{R}^n$ bounded, $C^1$, connected boundary, $\triangle u = 0$, $u$ bounded

\[
(S_{\partial \Omega}(\hat{n} \cdot \nabla u) - D_{\partial \Omega} u)(x) + (S_{\partial B_r}(\hat{n} \cdot \nabla u) - D_{\partial B_r} u)(x) = u(x)
\]

for $x$ between $\partial \Omega$ and $B_r$.

Now $r \to \infty$.

Behavior of individual terms?

Use mean value theorem and Gauss to estimate

\[
|\nabla u| \leq C/r.
\]
Theorem 15 (Green’s Formula in the exterior [Kress LIE 2nd ed. Thm 6.10])

\[
(S_{\partial \Omega}(\hat{n} \cdot \nabla u) - D_{\partial \Omega} u)(x) + \text{PV} u_{\infty} = u(x)
\]

for some constant \( u_{\infty} \). Only for \( n = 2 \),

\[
u_{\infty} = \frac{1}{2\pi r} \int_{|y|=r} u(y)ds_y.
\]

Theorem 16 (Green’s Formula in the exterior [Kress LIE 2nd ed. Thm 6.10])

\[
(S_{\partial \Omega}(\hat{n} \cdot \nabla u) - D_{\partial \Omega} u)(x) + u_{\infty} = u(x)
\]

Realize the power of this statement:

Can we use this to bound \( u \) as \( x \to \infty \)?
Consider the behavior of the fundamental solution as \( r \to \infty \).
How about $u$’s derivatives?
8 Boundary Value Problems

8.1 Laplace
## Boundary Value Problems: Overview

<table>
<thead>
<tr>
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<th>Dirichlet</th>
<th>Neumann</th>
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<tbody>
<tr>
<td><strong>Int.</strong></td>
<td>$\lim_{x \to \partial \Omega^-} u(x) = g$</td>
<td>$\lim_{x \to \partial \Omega^-} \hat{n} \cdot \nabla u(x) = g$</td>
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<td></td>
<td>+ unique</td>
<td>$\circ$ may differ by constant</td>
</tr>
<tr>
<td><strong>Ext.</strong></td>
<td>$\lim_{x \to \partial \Omega^+} u(x) = g$</td>
<td>$\lim_{x \to \partial \Omega^+} \hat{n} \cdot \nabla u(x) = g$</td>
</tr>
<tr>
<td></td>
<td>$u(x) = \begin{cases} O(1) &amp; 2D \text{ as }</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td>+ unique</td>
<td>+ unique</td>
</tr>
</tbody>
</table>

with $g \in C(\partial \Omega)$.

What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)

Dirichlet uniqueness: why?

Neumann uniqueness: why?

Truth in advertising: Missing assumptions on $\Omega$?
What’s a DtN map?

Next mission: Find IE representations for each.
Uniqueness of Integral Equation Solutions

Theorem 17 (Nullspaces [Kress LIE 2nd ed. Thm 6.20])

- \( N(I/2 - D) = \{0\} \)
- \( N(I/2 - S') = \{0\} \)
- \( N(I/2 + D) = \text{span}\{1\}, \quad N(I/2 + S') = \text{span}\{\psi\}, \quad \text{where} \int \psi \neq 0. \)

Show \( N(I/2 - D) = \{0\} \).

Show \( N(I/2 - S') = \{0\} \).

Show \( N(I/2 + D) = \text{span}\{1\} \).

What extra conditions on the RHS do we obtain?