TODAY

- hw5
- project sign-ups
- proposals

\[
\left( \frac{1}{2} - 0 \right) y = 0 \implies y = 0
\]

- D 1

0

1
Theorem 12 (Green’s Formula [Kress LIE 2nd ed. Thm 6.5])  If $\triangle u = 0$, then

\[
(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} 
    u(x) & x \in \Omega \\
    \frac{u(x)}{2} & x \in \partial\Omega \\
    0 & x \notin \Omega 
\end{cases}
\]

Suppose I know ‘Cauchy data’ $(u|_{\partial\Omega}, \hat{n} \cdot \nabla u|_{\partial\Omega})$ of $u$. What can I do?

What if $\Omega$ is an exterior domain?

What if $u = 1$? Do you see any practical uses of this?
Things harmonic functions (don’t) do

**Theorem 13 (Mean Value Theorem [Kress LIE 2nd ed. Thm 6.7])** If $\Delta u = 0$,

$$u(x) = \int_{B(x,r)} u(y) \, dy = \int_{\partial B(x,r)} u(y) \, dy$$

Define $\overline{\int}$?

$$\overline{\int_\Omega f(x) \, dx} = \frac{1}{|\Omega|} \int_{\overline{\Omega}} f(x) \, dx$$

Trace back to Green’s Formula (say, in 2D):

**Theorem 14 (Maximum Principle [Kress LIE 2nd ed. 6.9])** If $\Delta u = 0$ on compact set $\bar{\Omega}$:

$u$ attains its maximum on the boundary.
\[ u(x) = \nabla \log (|x|) - D(u) \]

\[ = \frac{1}{2\pi} \int_{\partial B(x,r)} \log (|x-y|) \cdot \nabla u(y) \, ds(y) - D(u) \]

\[ = \frac{1}{2\pi} \log (|x|) \int_{\partial B(x,r)} \nabla \cdot D(x, y) \cdot u(y) \, dy \]

\[ \nabla \log (|x|) = \frac{x}{|x|^2} \]

\[ = -\frac{1}{2\pi} \int_{\partial B(x,r)} \frac{r^2}{|r|^2} \cdot u(y) \, dy \]

\[ = -\frac{1}{2\pi r} \int_{\partial B(x,r)} u(y) \, dy \]
Suppose it were to attain its maximum somewhere inside an open set...

What do our *constructed* harmonic functions (i.e. layer potentials) do there?
Jump relations

Let $[X] = X_+ - X_-$. (Normal points towards $+" = \text{"exterior}".)

[Kress LIE 2nd ed. Thm. 6.14, 6.17,6.18]
\[ \lim_{x \to x_0 \pm} (S'\sigma) = \left(S' \pm \frac{1}{2}l\right) \sigma(x_0) \Rightarrow [S\sigma] = 0 \]
\[ \lim_{x \to x_0 \pm} (D\sigma) = \left(D \pm \frac{1}{2}l\right) \sigma(x_0) \Rightarrow [D\sigma] = \sigma, [D'\sigma] = 0 \]

Truth in advertising: Assumptions on \( \Gamma \)?

Sketch the proof for the single layer.

Sketch proof for the double layer.

\( \Theta \Omega \) is C

(construct a sequence of functions with \( \Theta \) of the singularity removed and show uniform convergence)
Represent a target point near the boundary

\[ x = z + h \hat{y}(z) \]

\[ x \in \Gamma \]

\[ \mathbf{D} \sigma(x) = \sigma(z) \mathbf{D} \mathbf{T}(x) + (\mathbf{D} \sigma - \mathbf{D} \sigma(z))(x) \]

\[ \int_{\Gamma} \hat{n}_y \cdot \nabla_y G(x,y) (\sigma(y) - \sigma(z)) \, ds(y) \]
Green’s Formula at Infinity (skipped)

$\Omega \subseteq \mathbb{R}^n$ bounded, $C^1$, connected boundary, $\triangle u = 0$, $u$ bounded

$$(S_{\partial \Omega}(\hat{n} \cdot \nabla u) - D_{\partial \Omega} u)(x) + (S_{\partial B_r}(\hat{n} \cdot \nabla u) - D_{\partial B_r} u)(x) = u(x)$$

for $x$ between $\partial \Omega$ and $B_r$.

Now $r \to \infty$.

Behavior of individual terms?

Use mean value theorem and Gauss to estimate

$$|\nabla u| \leq C/r.$$
Theorem 15 (Green’s Formula in the exterior [Kress LIE 2nd ed. Thm 6.10])

\[
\int_{\partial \Omega} \hat{n} \cdot \nabla u = 0 \quad (S_{\partial \Omega}(\hat{n} \cdot \nabla u) - D_{\partial \Omega} u)(x) + \text{PV} u_\infty = u(x)
\]

for some constant \( u_\infty \). Only for \( n = 2 \),

\[
u_\infty = \frac{1}{2\pi r} \int_{|y|=r} u(y) \, ds_y.
\]

Theorem 16 (Green’s Formula in the exterior [Kress LIE 2nd ed. Thm 6.10])

\[
(S_{\partial \Omega}(\hat{n} \cdot \nabla u) - D_{\partial \Omega} u)(x) + u_\infty = u(x)
\]

Realize the power of this statement:

Can we use this to bound \( u \) as \( x \to \infty \)?

Consider the behavior of the fundamental solution as \( r \to \infty \).
How about $u$'s derivatives?

\[ \frac{1}{\sqrt{2}} \]
8 Boundary Value Problems

8.1 Laplace
## Boundary Value Problems: Overview

<table>
<thead>
<tr>
<th>Dirichlet</th>
<th>Neumann</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Int.</strong></td>
<td></td>
</tr>
<tr>
<td>( \lim_{x \to \partial \Omega^-} u(x) = g )</td>
<td>( \lim_{x \to \partial \Omega^-} \hat{n} \cdot \nabla u(x) = g )</td>
</tr>
<tr>
<td>+ unique</td>
<td>+ may differ by constant</td>
</tr>
<tr>
<td><strong>Ext.</strong></td>
<td></td>
</tr>
<tr>
<td>( \lim_{x \to \partial \Omega^+} u(x) = g )</td>
<td>( \lim_{x \to \partial \Omega^+} \hat{n} \cdot \nabla u(x) = g )</td>
</tr>
<tr>
<td>( u(x) = \begin{cases} O(1) &amp; 2D \text{ as }</td>
<td>x</td>
</tr>
<tr>
<td>+ unique</td>
<td>+ unique</td>
</tr>
</tbody>
</table>

with \( g \in C(\partial \Omega) \).

What does \( f(x) = O(1) \) mean? (and \( f(x) = o(1) \)?)

Dirichlet uniqueness: why? \( \text{follows from the maximum principle} \)

Neumann uniqueness: why?

Truth in advertising: Missing assumptions on \( \Omega \)?
Neumann uniqueness:

\[ u_1, u_2 \text{ harmonic} \quad \partial_n u_1 = \partial_n u_2 \quad \tilde{u} = u_1 - u_2 \]

\[ \int \nabla \tilde{u} \cdot \nabla \tilde{u} = \int_{\partial \Omega} \tilde{u} (\tilde{u} \cdot \mathbf{n}) \, ds \]

\[ \int_{\Omega_1} \chi_{\omega_1} \nabla u_1 \cdot \nabla u_1 \, dx = \int_{\Omega_2} \chi_{\omega_2} \nabla u_2 \cdot \nabla u_2 \, dx \]
What’s a DtN map?

Next mission: Find IE representations for each.
Uniqueness of Integral Equation Solutions

Theorem 17 (Nullspaces [Kress LIE 2nd ed. Thm 6.20])

- $N(I/2 - D) = \{0\}$
- $N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \text{span}\{1\}$, $N(I/2 + S') = \text{span}\{\psi\}$, where $\int \psi \neq 0$.

Show $N(I/2 - D) = \{0\}$.

Show $N(I/2 - S') = \{0\}$.

Show $N(I/2 + D) = \text{span}\{1\}$.

What extra conditions on the RHS do we obtain?
→ “Clean” Existence for 3 out of 4.
Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}$...but we do not know $\psi$.

Use a different kernel:

$$\hat{n} \cdot \nabla_y G(x, y) \rightarrow \hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in \Omega$)

- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce $u = 0$ on exterior.
- $|x|^{n-2}u(x) = ?$ on exterior
- Thus $\int \phi = 0$. Contribution of the second term?
- $\phi/2 + D\phi = 0$, i.e. $\phi \in N(I/2 + D) =$?
• Existence/uniqueness?

→ Existence for 4 out of 4.

Remaining key shortcoming of IE theory for BVPs?
What's the problem? *(Hint: Jump condition for constant density)*

At corner $x_0$: (2D)

$$
\lim_{x \to x_0^\pm} = \int_{\partial\Omega} \hat{n} \cdot \nabla_y G(x, y) \phi(y) ds_y \pm \frac{1}{2} \frac{\text{opening angle on } \pm \text{ side}}{\pi} \phi
$$

→ non-continuous behavior of potential on $\Gamma$ at $x_0$

What space have we been living in?

Fixes:

- $l +$ Bounded (Neumann) + Compact (Fredholm)