Theorem 5 *$A : X \to X$ Banach, $\|A\| < 1$*

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

with $\|(I - A)^{-1}\| \leq 1/(1 - \|A\|)$.

- How does this rely on completeness/Banach-ness?
- There's an iterative procedure hidden in this.

  (Called '*Picard Iteration*'. Cf: Picard-Lindelöf theorem.)

  Hint: How would you compute $\sum_k A^k f$?
- **Q:** Why does this fall short?

  $\|A\| \leqslant 1$ is *way* to restrictive a condition.

  $\to$ We'll need better technology.

Biggest Q: If Cauchy sequences are too weak a tool to deliver a limit, where else are we going to get one?

6.2 Compactness

Compact sets

Definition 6 (Precompact/Relatively compact) *$M \subseteq X$ precompact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in X*

Definition 7 (Compact/'Sequentially complete') *$M \subseteq X$ compact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in M*

- Precompact $\Rightarrow$ bounded
- Precompact $\Leftrightarrow$ bounded (finite dim. only!)

Counterexample?

Looking for a bounded set where not every sequence contains a convergent subsequence.
$\rightarrow$ Make use of the fact that there are infinitely many 'directions' (dimensions).

Precompactness 'replaces' boundedness in ∞ dim (because boundedness is 'not strong enough')

Compact Operators

X, Y: Banach spaces

Definition 8 (Compact operator) $T : X \to Y$ *is* compact $:\Leftrightarrow$ $T($*bounded set*$)$ *is precompact.*

- T, S compact $\Rightarrow \alpha T + \beta S$ compact
- *One of* T, S compact $\Rightarrow S \circ T$ compact
- T_n all compact, $T_n \to T$ in operator norm $\Rightarrow T$ compact

Questions:

- Let $\dim T(X) < \infty$. Is T compact?
- Is the identity operator compact?

Intuition about Compact Operators

- Compact operator: As finite-dimensional as you're going to get in infinite dimensions.
- Not clear yet–but they are moral (∞-dim) equivalent of a matrix having *low numerical rank*.
- Are compact operators continuous (=bounded)?
- What do they do to high-frequency data?
- What do they do to low-frequency data?

Arzelà-Ascoli

Let $G \subset \mathbb{R}^n$ be compact.

> **Theorem 6 (Arzelà-Ascoli)** *$U \subset C(G)$ is precompact iff it is bounded and* equicontinuous.

Equicontinuous means

For all $x, y \in G$ $\quad \delta = \delta(x, \varepsilon, U)$

for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $f \in U$

if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Continuous means:

For all $x, y \in G$

for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\delta = \delta(x, \varepsilon, f)$$

$$\delta = \delta(\varepsilon, f)$$

Sketch of proof:

$U \subset C(G)$ precompact $\iff$ bdd + equicontinuous

"$\Longrightarrow$"

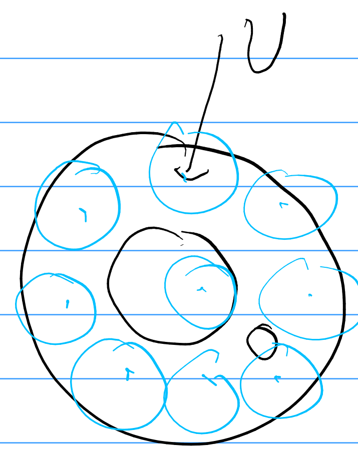
Since U is precompact

$\exists\ (f_i) \subseteq C(G)$ finite

s.t. $\forall f \in U\ \exists f_i$

$|f - f_i| < \varepsilon$

(total boundedness)



Let $\varepsilon > 0$. Let $x, y \in G$.

For every f pick an f_i such that $|f - f_i| < \varepsilon$

$< \varepsilon$ simultaneously $\forall i$

$$|f(x) - f(y)| < |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)|$$

$< 3\varepsilon$

6.3 Integral Operators

$\Leftarrow$ see book "diagonal argument"

Integral Operators are Compact

Theorem 7 (Continuous kernel $\Rightarrow$ compact [Kress LIE Thm. 2.21]) *$G \subset \mathbb{R}^m$ compact, $K \in C(G^2)$. Then*

$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

$: C(G) \to C(G)$

is compact on $C(G)$.

Let $U \subset C(G)$ be bounded

Use A-A. (a statement about compact *sets*) Show: $A(U)$ is compact

What is there to show?

Pick $U \subset C(G)$. $A(U)$ bounded?

Yes, because the operator is bounded. $\phi \in U$

$$|A\phi(x)| = \left|\int_G K(x,y)\phi(y)dy\right|$$

$$\le \underbrace{|\max K|\ |G|}_{\text{bound on } |A|}\ |\max \phi|$$

$A(U)$ equicontinuous?

Yes, because K *uniformly* continuous on $G \times G$ because $G \times G$ compact.

Let $\varepsilon > 0$. Let $x, y \in G$.

$$|A\phi(x) - A\phi(y)| = \left| \int_G K(x,z)\phi(z)\,dz - \int_G K(y,z)\phi(z)\,dz \right|$$

$$= \left| \int_G \big(K(x,z) - K(y,z)\big)\phi(z)\,dz \right|$$

Since K is uniformly continuous, pick $\delta > 0$ independent of $a, b \in G \times G$. If $|x-y| < \delta$ (in G) then $|(x,z)-(y,z)| < \delta$ in $G \times G$.

$$\le \varepsilon\, |G|\, |\max \phi| < \varepsilon\, |G|\, |U|$$

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition 9 (Weakly singular kernel)

- *K defined, continuous everywhere except at $x = y$*
- *There exist $C > 0$, $\alpha \in (0, n]$ such that*

$$|K(x, y)| \leq C|x - y|^{\alpha - n} \qquad (x \neq y)$$

Theorem 8 (Weakly singular kernel $\Rightarrow$ compact [Kress LIE Thm. 2.22]) *K weakly singular. Then*

$$(A\phi)(x) := \int_G K(x, y)\phi(y)dy.$$

is compact on $C(G)$.

Outline the proof.

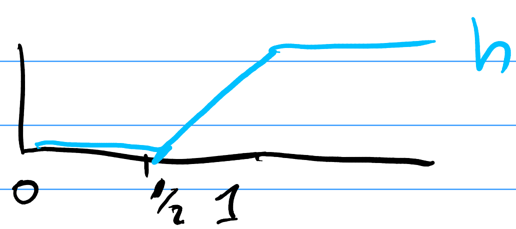
Let $\rho = |x-y|$. WLOG $x = 0$.

$$\left|\int_G K(x,y)\,\phi(y)\,dy\right|$$

$$\le \int_G |K(x,y)|\,|\phi(y)|\,dy$$

d ← domain is bounded

$$< |\phi_\infty|\, C\,\omega_n \int_0^d \rho^{\alpha-n}\rho^{n-1}\,d\rho \le |\phi|_\infty C\,\omega_n \frac{d^\alpha}{\alpha}$$



$$K_n(x,y) = \underbrace{h(n|x-y|)\,K(x,y)}_{\text{continuous kernel}}$$

Show: as $n \to \infty$

$$|A_n - A|_\infty \to 0$$

$$A_n\phi(x) = \int_G K_n(x,y)\,\phi(y)\,dy.$$

We have

$$|A_n\phi(x) - A\,\phi(x)|$$

$$\le \left|\int_G \big(K_n(x,y) - K(x,y)\big)\,\phi(y)\,dy\right|$$

$$< |\phi|_\infty\, C \int_0^{1/n} \rho^{\alpha-n}\rho^{n-1}\,d\rho$$

Weakly singular (on surfaces)

$\Omega \subset \mathbb{R}^n$ bounded, open, C^1

Definition 10 (Weakly singular kernel (on a surface))

- *K defined, continuous everywhere except at $x = y$*
- *There exist $C > 0$, $\alpha \in \underbrace{(0, n-1]}$ such that*

$$|K(x,y)| \le C|x-y|^{\underbrace{\alpha-n+1}} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

Theorem 9 (Weakly singular kernel ⇒ compact [Kress LIE Thm. 2.23]) *K weakly singular on $\partial\Omega$. Then*

$$(A\phi)(x) := \int_{\cancel{\partial\Omega}} K(x,y)\phi(y)dy.$$

$\partial\Omega$

is compact on $C(\cancel{\partial}\Omega)$. $C(\partial\Omega)$

Q: Has this estimate gotten worse or better?

6.4 Riesz and Fredholm

$$I - A \quad , \quad A \text{ compact}$$

$$\varphi - \int \underbrace{K}\, \varphi = f$$

Riesz Theory (I)

Still trying to solve

$$L\phi := (I - A)\phi = \phi - A\phi = f$$

with A compact.

Theorem 10 (First Riesz Theorem [Kress, Thm. 3.1]) *$N(L)$ is finite-dimensional.*

Questions:

- What is $N(L)$ again? = nullspace of $L = \{x \in X \mid x = Ax\}$
- Why is this good news?
- Show it.

Good news because each dimension in $N(L)$ is an obstacle to invertibility. Now we know that there's only 'finitely many obstacles'.

Proof:

- $N(L)$ closed. (Why?)

$N(L) = \{x \in X \mid x = Ax\}$

Take a sequence $(x_n) \subseteq N(L)$, $(x_n) \to y \in X$

By continuity $Ly = \lim_k Lx_k = 0$.

$\Rightarrow y \in N(L)$

- $L\phi = 0$ means what for A?
- When is the identity compact again?

On $N(L)$.

$Ax = x$

$A = I\big|_{N(L)}$.

$A\big|_{N(L)}$ is the identity.

The identity is compact $\Leftrightarrow$ space is finite dim.

A compact

$\Downarrow$

$N(L)$ is finite dimensional

Riesz Theory (Part II)

Theorem 11 (Riesz theory [Kress, Thm. 3.4]**)** *A compact.* **Then:**

- $(I - A)$ *injective* $\Leftrightarrow$ $(I - A)$ *surjective*
  - *It's either bijective or neither s nor i.*
- *If $(I - A)$ is bijective, $(I - A)^{-1}$ is bounded.*

completely general for bounded linear operators on a Banach space

Rephrase for solvability

If the solution to $(I - A)\varphi = 0$ is unique ($\varphi = 0$),then $(I - A)\varphi = f$ has a unique solution!

Main impact?

A real solvability result!

Key shortcoming?

Gives out completely if there happens to be a nullspace.

<u>Solvability</u>

$(1-A)$ injective $\Leftrightarrow$ $(1-A)$ surjective

$(1-A)$ injective

$\Updownarrow$

$(1-A)x=0 \Rightarrow x=0$

$\Updownarrow$

$\forall f \quad \exists x$

$(1-A)x=f$

Hilbert spaces

$(,) : X \times X \to \mathbb{C}$

Hilbert space: Banach space with a norm coming from an *inner product*:

$|x| = \sqrt{(x,x)}$

$(\alpha x + \beta y, z) = ?$ $(\alpha x, z) + (\beta y, z)$

$(x, \alpha y + \beta z) = ?$ $\overline{\alpha (y, x)} + \overline{\beta (z, x)}$

$(x, x)?$ $(x, x) \geq 0$, $= 0 \Leftrightarrow x = 0$

$(y, x) = ?$

> Is $C^0(G)$ a Hilbert space?

No, no inner product generates $\|\cdot\|_\infty$.

> Name a Hilbert space of functions.

$L^2(\Omega)$ with

$$(f, g) = \int_\Omega f \cdot g.$$