Theorem 5 \[ A : X \to X \text{ Banach, } \|A\| < 1 \]

\[ (I - A)^{-1} = \sum_{k=0}^{\infty} A^k \]

with \( \|(I - A)^{-1}\| \leq 1/(1 - \|A\|) \).

- How does this rely on completeness/Banach-ness?
- There’s an iterative procedure hidden in this.
  (Called ‘Picard Iteration’. Cf: Picard-Lindelöf theorem.)

  **Hint:** How would you compute \( \sum_k A^k f \)?

- **Q:** Why does this fall short?
  \( \|A\| \leq 1 \) is way to restrictive a condition.

  \[ \rightarrow \text{ We’ll need better technology.} \]

  **Biggest Q:** If Cauchy sequences are too weak a tool to deliver a limit, where else
are we going to get one?
6.2 Compactness
Compact sets

**Definition 6 (Precompact/Relatively compact)** \( M \subseteq X \) precompact: \( \iff \) all sequences \((x_k) \subseteq M\) contain a subsequence converging in \( X \)

**Definition 7 (Compact/‘Sequentially complete’)** \( M \subseteq X \) compact: \( \iff \) all sequences \((x_k) \subseteq M\) contain a subsequence converging in \( M \)

- Precompact \( \Rightarrow \) bounded
- Precompact \( \Leftrightarrow \) bounded (finite dim. only!)

Counterexample?

Looking for a bounded set where not every sequence contains a convergent subsequence. 
\( \rightarrow \) Make use of the fact that there are infinitely many ‘directions’ (dimensions).

Precompactness ‘replaces’ boundedness in \( \infty \) dim (because boundedness is ‘not strong enough’).
Compact Operators

X, Y: Banach spaces

**Definition 8 (Compact operator)**  
\( T : X \rightarrow Y \) is compact \( \iff \) \( T(\text{bounded set}) \) is precompact.

- \( T, S \) compact \( \Rightarrow \) \( \alpha T + \beta S \) compact
- \( \text{One of } T, S \) compact \( \Rightarrow \) \( S \circ T \) compact
- \( T_n \) all compact, \( T_n \rightarrow T \) in operator norm \( \Rightarrow \) \( T \) compact

Questions:

- Let \( \dim T(X) < \infty \). Is \( T \) compact?
- Is the identity operator compact?
Intuition about Compact Operators

• Compact operator: As finite-dimensional as you’re going to get in infinite dimensions.

• Not clear yet—but they are moral (∞-dim) equivalent of a matrix having low numerical rank.

• Are compact operators continuous (=bounded)?

• What do they do to high-frequency data?

• What do they do to low-frequency data?
**Arzelà-Ascoli**

Let $G \subset \mathbb{R}^n$ be compact.

**Theorem 6 (Arzelà-Ascoli)** $U \subset C(G)$ is precompact iff it is bounded and equicontinuous.

**Equicontinuous** means

For all $x, y \in G$

for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $f \in U$

if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

**Continuous** means:

For all $x, y \in G$

for all $\epsilon > 0$ there exists a $\delta > 0$ such that
\[ \delta = \delta (x, \varepsilon, f) \]

\[ \forall \xi, \eta \in f \exists \varepsilon > 0 \text{ such that } |f(x) - f(y)| < \varepsilon \]

Sketch of proof:

\[ U \subset C(B) \text{ precompact } \iff \text{bdd} + \text{equicontinuous} \]

Since \( U \) is precompact,

\[ \exists f \in C(B) \text{ finite } \]

\[ \forall \xi, \eta \in U \exists f \in \text{finite} \]

\[ |f(x) - f(y)| < \varepsilon \]

(Holomorphic boundedness)

Let \( \varepsilon > 0 \). Let \( x, y \in B \).

For every \( f \) pick \( \xi, \eta \in U \) such that

\[ |f(x) - f(y)| < \varepsilon \]

\[ |f(x) - f(y)| < |f(x) - f(\xi)| + |f(\xi) - f(\eta)| + |f(\eta) - f(y)| \leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \]

\[ \leq 3 \varepsilon \]
6.3 Integral Operators

“see book "diagonal argument"
**Theorem 7 (Continuous kernel ⇒ compact)** \( \text{[Kress LIE Thm. 2.21]} \quad G \subset \mathbb{R}^m \) compact, \( K \in C(G^2) \). Then

\[
(A\phi)(x) := \int_G K(x, y)\phi(y)dy.
\]

is compact on \( C(G) \).

Let \( U \subset C(G) \) be bounded.

Use A-A. (a statement about compact sets)

Show: \( A(U) \) is compact

What is there to show?

Pick \( U \subset C(G) \). \( A(U) \) bounded?

Yes, because the operator is bounded.

\[
|A\phi(x)| = \left| \int_G K(x, y)\phi(y)dy \right|
\]
Yes, because $K$ uniformly continuous on $G \times G$ because $G \times G$ compact.

Let $\varepsilon > 0$. Let $x, y \in G$.

\[
|A \phi(x) - A \phi(y)| = \left| \int_G K(x, z) \phi(z) \, dz - \int_G K(y, z) \phi(z) \, dz \right|
\]

\[
= \left| \int_G (K(x, z) - K(y, z)) \phi(z) \, dz \right|
\]

Since $K$ is uniformly continuous, pick $\delta > 0$

independent of $a, b \in G \times G$. If $|x - y| < \delta$ (i.e., $G$) then $|(x, z) - (y, z)| < \delta$ in $G \times G$.

\[
\leq \varepsilon \leq |G| |\max \phi| \cdot 3 \leq |G| |\max \phi| \cdot \varepsilon \]

\[
\leq |G| |\max \phi| \cdot \varepsilon \]

\[
\leq |G| |\max \phi| \cdot \varepsilon
\]
Weakly singular

$G \subset \mathbb{R}^n$ compact

**Definition 9 (Weakly singular kernel)**
- $K$ defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n]$ such that
  \[ |K(x, y)| \leq C|x - y|^{{\alpha - n}} \quad (x \neq y) \]

**Theorem 8 (Weakly singular kernel $\Rightarrow$ compact [Kress LIE Thm. 2.22])**

If $K$ weakly singular. Then

\[(A\phi)(x) := \int_G K(x, y)\phi(y)dy.\]

is compact on $C(G)$.

Outline the proof.
Let \( p = |x - y| \). WLOG \( x = 0 \).

\[
\left| \int_6 \kappa(x, y) \phi(y) \, dy \right| \\
\leq \int_6 \left| \kappa(x, y) \right| |\phi(y)| \, dy \\
< \|\phi\|_{\infty} C_w \int_0^1 \rho^\alpha \, d\rho < \|\phi\|_{\infty} C_w \int_0^1 \frac{d\rho}{\alpha}
\]

\[
k_n(x, y) = \frac{h(n|x - y|)}{k(x, y)}
\]

Show: as \( n \to \infty \)

\[
|A_n - A|_{\infty} \to 0
\]

\[
A_n \phi(x) = \int_6 \kappa_n(x, y) \phi(y) \, dy
\]

We have

\[
|A_n \phi(x) - A \phi(x)| \\
\leq \left| \int_6 \left( \kappa_n(x, y) - \kappa(x, y) \right) \phi(y) \, dy \right| \\
< 1 \|\phi\|_{\infty} C \int_0^1 \rho^{\alpha-n} \rho^{n-1} \, d\rho
\]
Weakly singular (on surfaces)

\[ \Omega \subset \mathbb{R}^n \text{ bounded, open, } C^1 \]

**Definition 10 (Weakly singular kernel (on a surface))**

- \( K \) defined, continuous everywhere except at \( x = y \)
- There exist \( C > 0, \alpha \in (0, n - 1] \) such that

\[ |K(x, y)| \leq C|x - y|^{\alpha - n + 1} \quad (x, y \in \partial \Omega, \ x \neq y) \]

**Theorem 9 (Weakly singular kernel \( \Rightarrow \) compact [Kress LIE Thm. 2.23])** \( K \) weakly singular on \( \partial \Omega \). Then

\[ (A\phi)(x) := \int_{\partial \Omega} K(x, y)\phi(y)dy. \]

is compact on \( C(\partial \Omega) \).
Q: Has this estimate gotten worse or better?
6.4 Riesz and Fredholm

\[ (I - A), \quad A \text{ compact} \]

\[ \psi - \int \kappa \psi = f \]
Riesz Theory (I)

Still trying to solve

\[ L\phi := (I - A)\phi = \phi - A\phi = f \]

with \( A \) compact.

**Theorem 10 (First Riesz Theorem [Kress, Thm. 3.1])**

\( N(L) \) is finite-dimensional.

Questions:
- What is \( N(L) \) again?
- Why is this good news?
- Show it.

Good news because each dimension in \( N(L) \) is an obstacle to invertibility. Now we know that there’s only ‘finitely many obstacles’.

Proof:

- \( N(L) \) closed. (Why?)

\[ N(L) = \{ x \in X \mid x = Ax \} \]

Take a sequence \( (x_n) \subseteq N(L) \), \( (x_n) \to y \in X \)
By continuity \( L.y = \lim_{k \to \infty} (Lx_k) = 0 \), so \( y \in N(L) \).

- \( L\phi = 0 \) means what for \( A \)?
- When is the identity compact again?

\[ \downarrow \]

A \( | N(L) \) is the identity.

The identity is compact \( \iff \) span is finite dim.

A compact \( \downarrow \)

\( N(L) \) is finite dimensional.
Riesz Theory (Part II)

**Theorem 11 (Riesz theory [Kress, Thm. 3.4])** A compact. Then:

- $(I - A)$ injective $\iff$ $(I - A)$ surjective
  - It’s either bijective or neither s nor i.
- If $(I - A)$ is bijective, $(I - A)^{-1}$ is bounded.

Rephrase for solvability

If the solution to $(I - A)\varphi = 0$ is unique ($\varphi = 0$), then $(I - A)\varphi = f$ has a unique solution!

Main impact?

A real solvability result!

Key shortcoming?
Gives out completely if there happens to be a nullspace.

\[
\text{Solvability}
\]

\[
(1-A) \text{ injective } \Rightarrow (1-A) \text{ surjective}
\]

\[
(1-A) \text{ injective} \quad \exists \quad \forall \ f \ \exists \ x
\]

\[
(1-A)x = 0 \Rightarrow x = 0 
\]

\[
(1-A)x = f
\]
Hilbert spaces

Hilbert space: Banach space with a norm coming from an *inner product*:

\[
\|x\| = \sqrt{(x, x)}
\]

\[
(\alpha x + \beta y, z) =? \quad (\alpha (x, z) + (\beta y, z))
\]

\[
(x, \alpha y + \beta z) =? \quad \alpha (y, x) + \beta (z, x)
\]

\[
(x, x)? \quad (y, x) = 0, \quad \Rightarrow \quad x = 0
\]

\[
(y, x) =?
\]

Is \(C^0(G)\) a Hilbert space?

No, no inner product generates \(\|\cdot\|_\infty\).

Name a Hilbert space of functions.

\(L^2(\Omega)\) with

\[
(f, g) = \int_{\Omega} f \cdot g.
\]