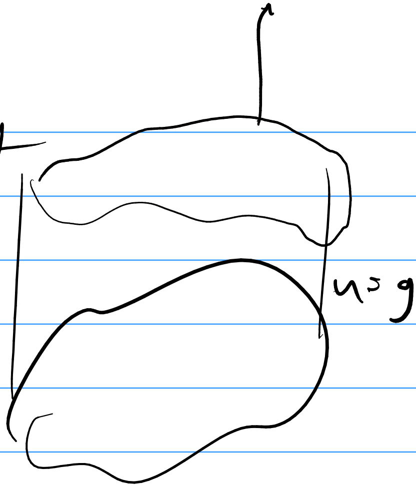
Today



$\Delta u = 0$

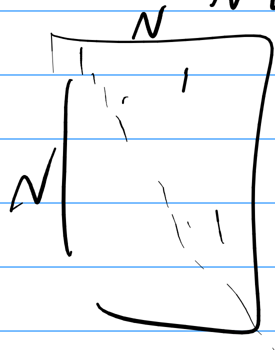
$N \times N \qquad k \ll N$

$N^{\alpha} k^{\beta}$

right now: $N^2 k$

$N k^2$

- RRQR
- ID



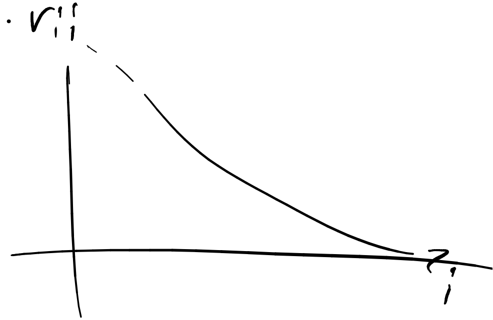
Rank-revealing/pivoted QR

Sometimes the SVD is too *good* (aka expensive)–we may need less accuracy/weaker promises, for a significant decrease in cost.

$$A\Pi = QR$$

$$= Q\left(\begin{array}{c|c} R_{11} & R_{12} \\ \hline & R_{22} \end{array}\right)$$

$$\sigma_{k+1} \le \|R_{22}\|_2$$

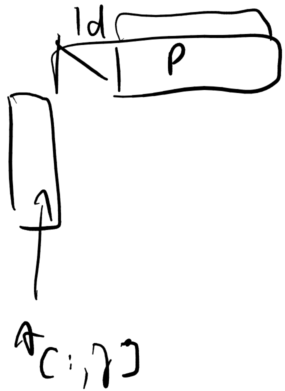


Interpolative Decomposition (ID)

Sometimes it would be helpful to know *which columns of A* contribute ‘the most’ to the rank.
(rather than have the waters muddied by an orthogonal transformation like in QR)

$$A \approx A_{[:,J]} P$$

$$J \in \mathbb{N}^k$$



RRQR

$\hookrightarrow A = Q \begin{pmatrix} R_{11} & R_{12} \end{pmatrix}$

Set $B = Q R_{11}$

$$P = \begin{pmatrix} \underbrace{\mathrm{Id}}_{k} & R_{11}^{-1} R_{12} \end{pmatrix}.$$

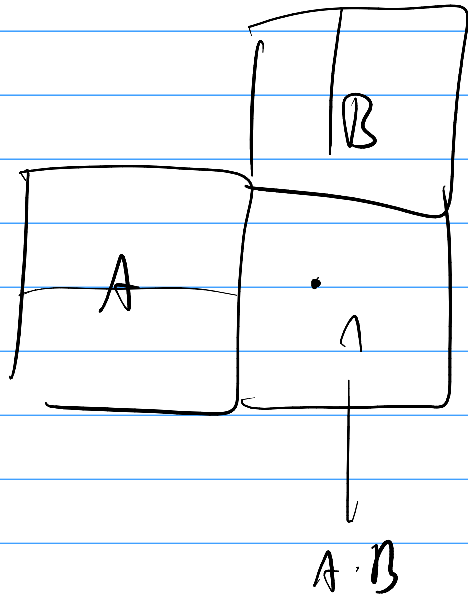
$$BP = Q R_{11} \begin{pmatrix} \mathrm{Id} & R_{11}^{-1} R_{12} \end{pmatrix}$$

$$= Q \begin{pmatrix} R_{11} & R_{12} \end{pmatrix} \approx A$$

$\hookrightarrow$

- P is well-cond
- entries $|\cdot| \leq 2$.

$A \cdot B$



What does the ID buy us?

> Specifically: Name a property that the ID has that other factorizations do not have.

$$A \approx Q Q^T A$$

column ID

$$A = A_{(:,J)} P$$

Row ID:

$$Q = P \; Q_{(J,:)}$$

$$A_{(J,:)} \approx \left[Q Q^T A \right]_{(J,:)}$$

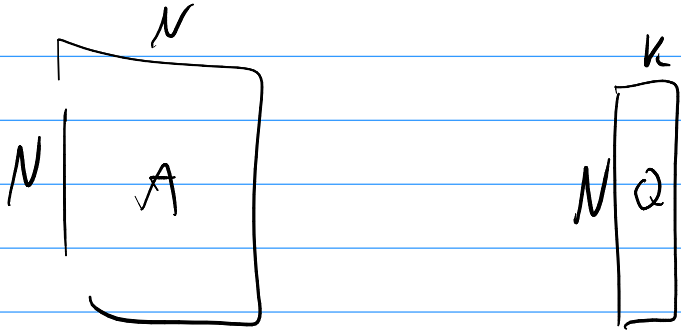
$$\approx Q_{(J,:)} Q^T A$$

$$= \left[P \, Q_{(J,:)} \right]_{(J,:)} Q^T A$$

$$= \underbrace{P_{[j,:]}}_{\text{Id}} Q_{[j,:]} Q^T A$$

$$= Q_{[j,:]} Q^\dagger A$$

$$\approx A$$



ID Q vs ID A

What does row selection mean for the LRA?

Leveraging the ID

Build a low-rank SVD with row extraction.

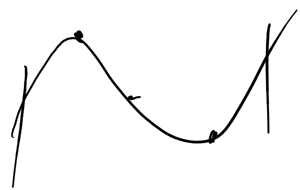
1. Obtain row subset J and upsampler P.
2. Compute a row QR

$$\underbrace{\left(A_{[J,:]}\right)^T}_{N\times k} = \underbrace{\bar{Q}}_{N\times k}\ \underbrace{\bar{R}}_{k\times k}$$

3. Upsample row coefficients: $\bar{R}^T$

$$\underbrace{Z}_{N\times k} = \underbrace{P}_{N\times k}\ \underbrace{\bar{R}^T}_{k\times k}$$

4. SVD $Z = U \Sigma \bar{V}^T$



N [ ] k

$N^2 k \leftarrow$ bad

$N k^2$

$$
\begin{aligned}
& U \Sigma (\tilde{Q} \tilde{V})^T \\
&= \underbrace{U \Sigma \tilde{V}^T}\, \tilde{Q}^T \\
&= Z \tilde{Q}^T \\
&= P \tilde{R}^T \tilde{Q}^T \\
&= P A_{(J,:)} \\
&\approx A
\end{aligned}
$$

Where are we now?

3 Rank and Smoothness

Punchline

What do (numerical) rank and smoothness have to do with each other?

Recap: Multivariate Taylor

How does Taylor's theorem get generalized to multiple dimensions?