Today:

Recap SVD

Connect low rank
to functions

HW1 graded HW2

$N \times N$ k

Nk^2

$QQ^* A$

ID Q vs ID A

What does row selection mean for the LRA?

$A \approx Q \underbrace{Q^T A}$,

$Q \approx P \, Q_{(J,:)}$

(same P) $A \approx P A_{(J,:)}$

$A = P \overbrace{A_{(J,:)}}$

Nk^2

$\stackrel{?}{=} \underline{P U \Sigma V^T}$

Could we SVD $PA_{(J,:)}$?

$\rightarrow$ No, that's all of A, so N^3.

$$A \approx Q Q^{\dagger} A$$

$$A_{(J,:)} \leftarrow$$

$$Q \approx P\, Q_{[J,:]}$$

k

N

$$(A_{(J,:)})^T = \bar{Q}\bar{R} \leftarrow Nk^2$$

$$P\bar{R}^T = U\Sigma\bar{V}^T \in Nk^2$$

$C \cdot n^2$ $\qquad\qquad$ $c \cdot n^3$

Strassen

$C \cdot n^{2.8}$ $\qquad\qquad$ $c n^3$

Storage: $\quad n^2$

Operations: $\quad n^3$

$$U \Sigma \overbrace{\left(\bar{Q} \, \bar{V} \right)}^{V}{}^{T} \leftarrow$$

$$= U \Sigma \bar{V}^T \breve{Q}^T$$

$$= P \breve{R}^{+} \breve{Q}^T$$

$$= P A_{(J,I)} \approx A$$

$$Ax = b \qquad A = U \Sigma V^T$$

$$x = V \Sigma^{-1} U^T b$$

$$V \Sigma^+ U^T$$

$$\left\| A x - b \right\|_2 \qquad \left\| A x - b \right\|_2$$

$$\min |x|_2$$

Leveraging the ID

Build a low-rank SVD with row extraction.

Where are we now?

3 Rank and Smoothness

Punchline

What do (numerical) rank and smoothness have to do with each other?

$$y(x) = \sum G_i(x)\, \sigma_i$$

$$G(x, y) = \frac{1}{\|x - y\|_2}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Recap: Multivariate Taylor

How does Taylor's theorem get generalized to multiple dimensions?

$$f(c+h) \approx \sum_{i=0}^{k} \frac{f^{(i)}(c)}{i!} h^i$$

Suppose: $\left| \frac{f^{(i)}(c)}{i!} h^i \right| \le \alpha^i$

$$\left| f(c+h) - \sum_{i=0}^{k} \frac{f^{(i)}(c)}{i!} h^i \right| = \left| \sum_{i=k+1}^{\infty} \frac{f^{(i)}(c)}{i!} h^i \right|$$

$$\le \sum_{i=k+1}^{\infty} \alpha^i \le \frac{1}{1-\alpha} \alpha^{k+1}$$

$\nu = (5, 3) \in 2D$

$\nu = (\nu_1, \dots, \nu_d)$

$|\nu| = \nu_1 + \dots + \nu_d$

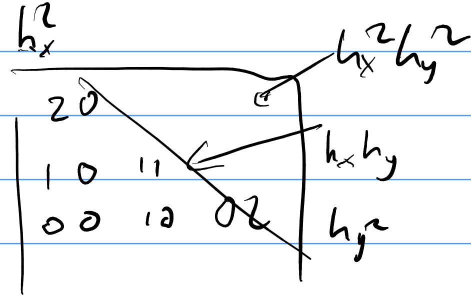
$\vec{x}^{\nu} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}^{\nu} = x_1^{\nu_1} \cdots x_d^{\nu_d}$

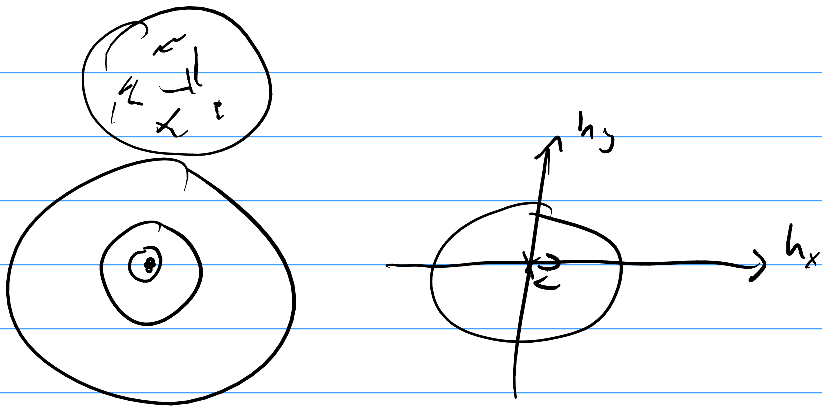
$\nu! = \nu_1! \cdots \nu_d!$

$D^{\nu} f = \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \cdots \frac{\partial^{\nu_d}}{\partial x_d^{\nu_d}} f$

1D: $f(c+h) \approx \sum_{i=0}^{k} \frac{f^{(i)}(c)}{i!} h^i$

nD: $f(\vec{c} + \vec{h}) \approx \sum_{|\nu| \le k} \frac{D^{\nu} f(\vec{c})}{\nu!} h^{\nu}$





h_y
h_x

Connect Taylor and Low Rank

Can Taylor help us establish low rank of an interaction?

Taylor on Potentials

Compute a Taylor expansion of a 2D Laplace point potential.

Local expansions as a Computational Tool

Low rank makes evaluating interactions cheap(er). Do local expansions help with that goal?

Taylor on Potentials, Again

Stare at that Taylor formula again.