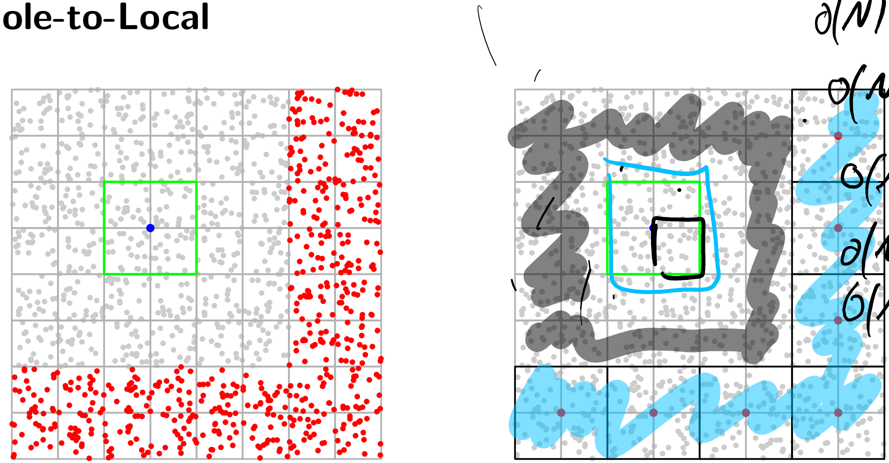
TODAY

Fast Multipole

Direct solving

PDE

Using Multipole-to-Local



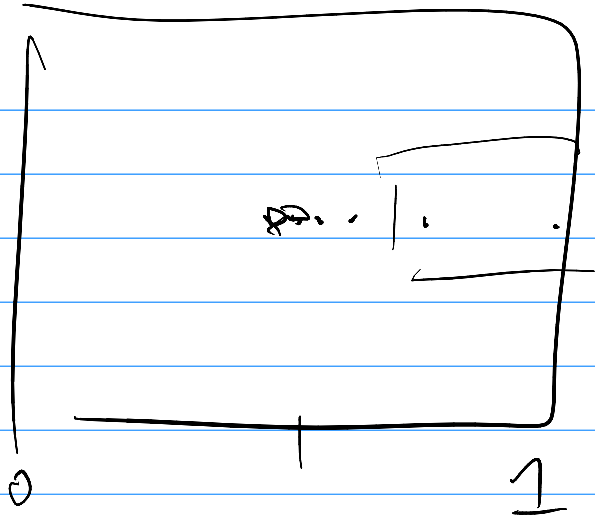
O(N) Form mpoles
O(N) Mpoles upw.
O(N) Locals downward
O(N) Eval local
O(N) Take care of near interaction

(Figure credit: G. Martinsson, Boulder)

Come up with an algorithm that computes the interaction in the figure.

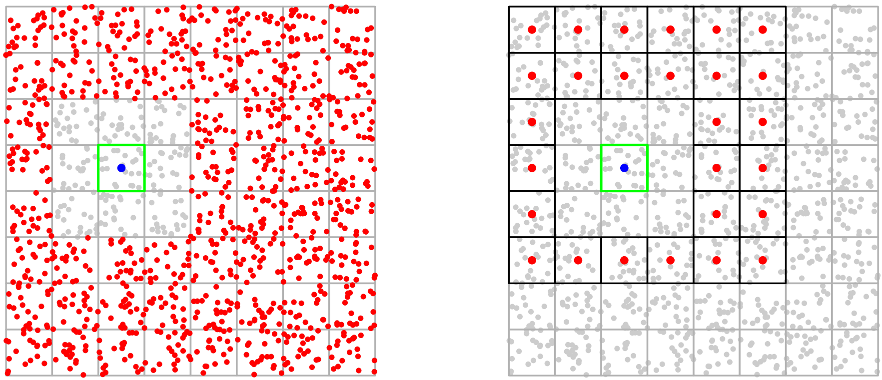
Locals downward:
for level = 0 ... L
for each box on level L:
M2L from "glorious 27"
Translate local from parent

Error: $\left(\frac{\sqrt{2}}{3}\right)^{p+1} = \left(\frac{d f_s}{d c t}\right)^{p+1}$ O(N)



0
1

Using Multipole-to-Local



(Figure credit: G. Martinsson, Boulder)

Assuming we retain information from the previous level, how can we obtain a valid local expansion on the target box?

Define ‘Interaction List’

For a box b, the interaction list I_b consists of all boxes b' so that

The Fast Multipole Method ('FMM')

Upward pass

1. Build tree
2. Compute interaction lists
3. Compute lowest-level multipoles from sources
4. Loop over levels $\ell = L-1, \ldots, 2$:
   (a) Compute multipoles at level ℓ by mp $\to$ mp

Downward pass

1. Loop over levels $\ell = 2, 3, \ldots, L-1$:
   (a) Loop over boxes b on level ℓ:
      i. Add contrib from I_b to local expansion by mp $\to$ loc
      ii. Add contrib from parent to local exp by loc $\to$ loc
2. Evaluate local expansion and direct contrib from 9 neighbors.

Overall algorithm: Now $O(N)$ complexity.

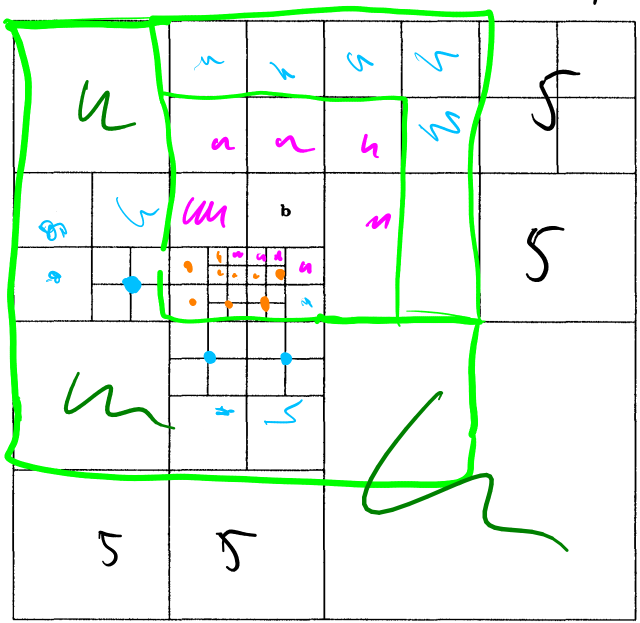
Note: L levels, numbered $0, \ldots, L-1$. Loop indices above *inclusive.*

What about adaptivity?

- direct ← List 1 ←
- Local from mpole
- mpole→tgt (smaller) ← List 3
- src→local (bigger) ← List 4

Local from pmnt ← List 5

List U



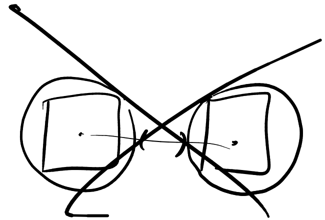


Figure credit: Carrier et al. (‘88)

What changes?

What about adaptivity?

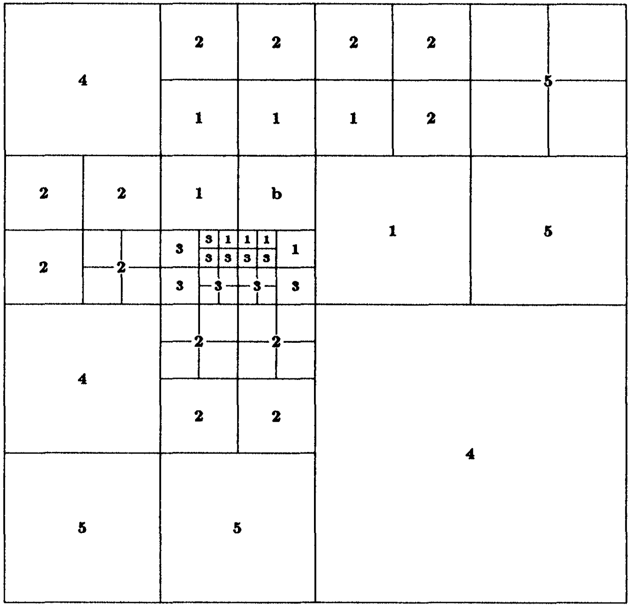


Figure credit: Carrier et al. ('88)

Make a list of cases:

What about solving?

Likely computational goal: Solve a linear system $Ax = b$. How do our methods help with that?

A Matrix View of Low-Rank Interaction

Only *parts of the matrix are low-rank*! What does this look like from a matrix perspective?

Block-separable matrices

How do we represent the low-rank structure of a matrix like this?

$$A = \begin{pmatrix} D_1 & A_{12} & A_{13} & A_{14} \\ A_{21} & D_2 & A_{23} & A_{24} \\ A_{31} & A_{32} & D_3 & A_{34} \\ A_{41} & A_{42} & A_{43} & D_4 \end{pmatrix}$$

where A_{ij} has low-rank structure?

Block-Separable Matrices

A *block-separable matrix* looks like this:

$$A = \begin{pmatrix} D_1 & P_1\tilde{A}_{12}\Pi_2 & P_1\tilde{A}_{13}\Pi_3 & P_1\tilde{A}_{14}\Pi_4 \\ P_2\tilde{A}_{21}\Pi_1 & D_2 & P_2\tilde{A}_{23}\Pi_3 & P_2\tilde{A}_{24}\Pi_4 \\ P_3\tilde{A}_{31}\Pi_1 & P_3\tilde{A}_{32}\Pi_2 & D_3 & P_3\tilde{A}_{34}\Pi_4 \\ P_4\tilde{A}_{41}\Pi_1 & P_4\tilde{A}_{42}\Pi_2 & P_4\tilde{A}_{43}\Pi_3 & D_4 \end{pmatrix}$$

Here:

- $\tilde{A}_{ij}$ smaller than A_{ij}
- D_i has full rank (not necessarily diagonal)
- P_i shared for entire row
- Π_i shared for entire column

Q: Why is it called that?