TODAY

- norms / Neumann series
- compactness
- Riesz theory

$$\left(\frac{I}{2} + D\right)\varphi = g$$

?

Operators

X, Y: Banach spaces

$A : X \to Y$ linear operator

Definition 2 (Operator norm) $\|A\| := \sup\{\|Ax\| : x \in X, \|x\| = 1\}$

Theorem 1 $\|A\|$ *bounded* $\Leftrightarrow$ A *continuous*

Operators: Examples

Which of these is bounded as an operator on functions on the real line?

- Multiplication by a scalar
- "Left shift"
- Fourier transform
- Differentiation
- Integration
- Integral operators

$f \mapsto \alpha f$

$A: f(x) \mapsto f(x+\alpha)$

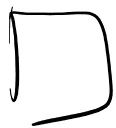
$A(af+bg) = af(x+\alpha) + bg(x+\alpha)$

$C^0[0,2\pi]$

$\|\cdot\|_\infty$

$\|f\|_\infty + \|f'\|_\infty$

$\|f\| = \int |f| \cdot g \, dx$



Norm equivalent: $c\|x\| \leq \|x\|' \leq C\|x\|$

Integral Operators: Zoology

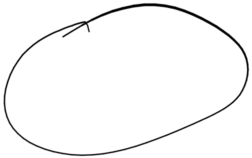
Volterra	Fredholm
$\int_a^x k(x,y)f(y)dy = g(x)$	$\int_G k(x,y)f(y)dy = g(x)$

First kind	Second kind
$\int_G k(x,y)f(y)dy = g(x)$	$f(x) + \int_G k(x,y)f(y)dy = g(x)$

$k + \delta$

$(I + A) f$

Important assumption: kernel smoothness



Connections to Complex Variables

Complex analysis is *full* of integral operators:

- Cauchy's integral formula:

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{z-a} f(z)\, dz$$

- Cauchy's differentiation formula:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{1}{(z-a)^{n+1}} f(z)\, dz$$

Integral Operators: Boundedness (=Continuity)

Theorem 2 (Continuous kernel ⇒ bounded) *$G \subset \mathbb{R}^n$ closed, bounded ("compact"), $K \in C(G^2)$. Let*

$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

Then

$$\|A\|_\infty = \max_{x\in G} \int_G |K(x,y)|dy.$$

Show '⩽'.

Let φ have $\|\varphi\|_\infty = 1$ and $\varphi \in C^0(G)$.

$$\|A\varphi\| = \left\| \int_G k(x,y)\varphi(y)dy \right\|_\infty$$

$$\leq \left\| \int_G k(x,y)dy \right\|_\infty$$

Solving Integral Equations

Given

$$(A\phi)(x) := \int_G K(x,y)\varphi(y)dy,$$

are we allowed to ask for a solution of

$$(\mathrm{Id} - A)\varphi = g?$$

$$\varphi = (\mathrm{Id} - A)^{-1} g$$

$$\left(\sum_{i=0}^{\infty} A^i\right) g$$

$$\frac{1}{1-A}$$

turn this into an algorithm: Picard iteration

For this to make sense, have to have to

$$\|A\| < 1$$

Attempt 1: The Neumann series

Want to solve

$$\varphi - A\varphi = (I - A)\varphi = g.$$

Formally:

$$\varphi = (I - A)^{-1} g.$$

What does that remind you of?

6.2 Compactness

Compact sets

Definition 3 (Precompact/Relatively compact) *$M \subseteq X$ precompact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in X*

Definition 4 (Compact/'Sequentially complete') *$M \subseteq X$ compact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in M*

- Precompact $\Rightarrow$ bounded
- Precompact $\Leftrightarrow$ bounded (finite dim. only!)

Counterexample?

c_0, sequence =

(1, 0, 0, ...)
(0, 1, 0, 0, 0 ...)
(0, 0, 1, 0, ...)

take $n \to \infty$

Compact Operators

$T(f_i) = \frac{1}{i} f_i$

X, Y: Banach spaces

Definition 5 (Compact operator) $T : X \to Y$ *is* compact $:\Leftrightarrow$ $T($*bounded set*$)$ *is precompact.*

- T, S compact $\Rightarrow \alpha T + \beta S$ compact
- *One of* T, S compact $\Rightarrow S \circ T$ compact
- T_n all compact, $T_n \to T$ in operator norm $\Rightarrow T$ compact

Questions:

- Let $\dim T(X) < \infty$. Is T compact?
- Is the identity operator compact?

Intuition about Compact Operators

- Compact operator: As finite-dimensional as you're going to get in infinite dimensions.
- Not clear yet–but they are moral (∞-dim) equivalent of a matrix having *low numerical rank*.
- Are compact operators continuous (=bounded)?
- What do they do to high-frequency data?
- What do they do to low-frequency data?

Arzelà-Ascoli

Let $G \subset \mathbb{R}^n$ be compact.

> **Theorem 3 (Arzelà-Ascoli)** *$U \subset C(G)$ is precompact iff it is bounded and* equicontinuous*.*

Equicontinuous means

For all $x, y \in G$

for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $f \in U$

if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Continuous means:

For all $x, y \in G$

for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x-y|<\delta$ then $|f(x)-f(y)|<\varepsilon$.

Uniformly continuous:

if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Intuition?

"Uniformly continuous"?

When does *uniform continuity* happen?

6.3 Integral Operators

Integral Operators are Compact

Theorem 4 (Continuous kernel ⇒ compact [Kress LIE Thm. 2.21]) *$G \subset \mathbb{R}^m$ compact, $K \in C(G^2)$. Then*

$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

is compact on $C(G)$.

Use A-A. (a statement about compact *sets*)

What is there to show?

Pick $U \subset C(G)$. $A(U)$ bounded?

$A(U)$ equicontinuous?

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition 6 (Weakly singular kernel)

- *K defined, continuous everywhere except at $x = y$*
- *There exist $C > 0$, $\alpha \in (0, n]$ such that*

$$|K(x, y)| \leq C|x - y|^{\alpha - n} \qquad (x \neq y)$$

Theorem 5 (Weakly singular kernel $\Rightarrow$ compact [Kress LIE Thm. 2.22]) *K weakly singular. Then*

$$(A\phi)(x) := \int_G K(x, y)\phi(y)dy.$$

is compact on $C(G)$.

Outline the proof.

Weakly singular (on surfaces)

$\Omega \subset \mathbb{R}^n$ bounded, open, C^1

Definition 7 (Weakly singular kernel (on a surface))

- *K defined, continuous everywhere except at $x = y$*
- *There exist $C > 0$, $\alpha \in (0, n-1]$ such that*

$$|K(x,y)| \leq C|x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

Theorem 6 (Weakly singular kernel $\Rightarrow$ compact [Kress LIE Thm. 2.23]) *K weakly singular on $\partial\Omega$. Then*

$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

is compact on $C(G)$.