Riesz Theory (Part II)

Theorem 8 (Riesz theory [Kress, Thm. 3.4]) *A compact.* **Then:**

- $(I - A)$ *injective* $\Leftrightarrow$ $(I - A)$ *surjective*
  - *It's either bijective or neither s nor i.*
- *If* $(I - A)$ *is bijective,* $(I - A)^{-1}$ *is bounded.*

Rephrase for solvability

Main impact?

Key shortcoming?

$$|f| = \sqrt{(f,f)}$$

$\alpha(x,y) + \beta(y,z) =$

$(x,y) = \overline{(y,x)}$

$(x,x) > 0 \iff x \neq 0$

(i) linear

(ii) antilinear

(iii) pos. definite

Examples of Hilbert spaces

$\mathbb{R}^n$ with $\sum x_i y_i = \langle x,y \rangle$

$L^2(G)$ with $\int f \bar{g}\, dx = \langle f, g \rangle$

$H^1(G)$ with $\int f\bar{g} + f'\bar{g}'\, dx = \langle f,g \rangle$

Is C^0 Hilbert space? No.

$|\cdot|_{C^0}$ Doesn't satisfy parallelogram law

$$2|x|^2 + 2|y|^2 = |x+y|^2 - |x-y|^2$$

Is C^0 equivalent to a Hilbert space?

$\exists\, c, C > 0$

$$c|f|_{C^0} < |f|_{L^2} < C|f|_{C^0}$$

fails

holds on compact G

Fact: Compactness results for $C(G)$ are also true for $L^2(G)$.

Cauchy-Schwartz

$$|(f,g)| < |f|_2\ |g|_2$$

$A : X \to X$

Adjoint Operators

$; X \to X$

Definition 8 (Adjoint oeprator) *A^* called adjoint to A if*

$$(Ax, y) = (x, A^* y)$$

for all x, y.

Facts:

- A^* unique
- A^* exists
- A^* linear
- A bounded $\Rightarrow$ A^* bounded
- A compact $\Rightarrow$ A^* compact

What is the adjoint operator in finite dimensions? (in matrix representation)

$(Ax, y) = y^T A x = (x, A^T y) =, \quad y^T (A^T)^T x$

$$A\phi(x) = \int K(x,y)\,\phi(y)\,dy$$

What do you expect to happen with integral operators? (on L^2)

Adjoint of the single-layer?

Adjoint of the double-layer? $K(x,y) = \partial_{n_y} \log|x-y|$

$K^*(x,y) = \partial_{n_x} \log|x-y|$

$$(Af, g) = \int \left(\int K(x,y) f(y)\,dy \right) g(x)\,dx$$

Define

$K(x,y) = \log|x-y|$

$K^*(x,y) = \log|x-y|$

$$K^*(x,y) = K(y,x)$$

$$= \int \int K^*(y,x)\, g(x)\, f(y)\,dy\,dx$$

$$= \iint K^*(y,x)\, g(x)\,dx\, f(y)\,dy$$

$$= (f, A^* g)$$

]

Solvability

$$(I - A)x = f$$

$$(I - A^*)x = f$$

Either both equations are always solvable, or there are finitely many functions $f_1 \cdots f_n \in N(I-A)$ s.t.

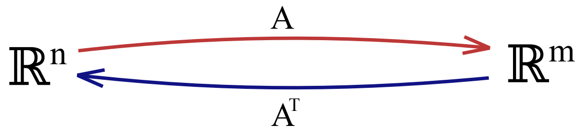
the solution to

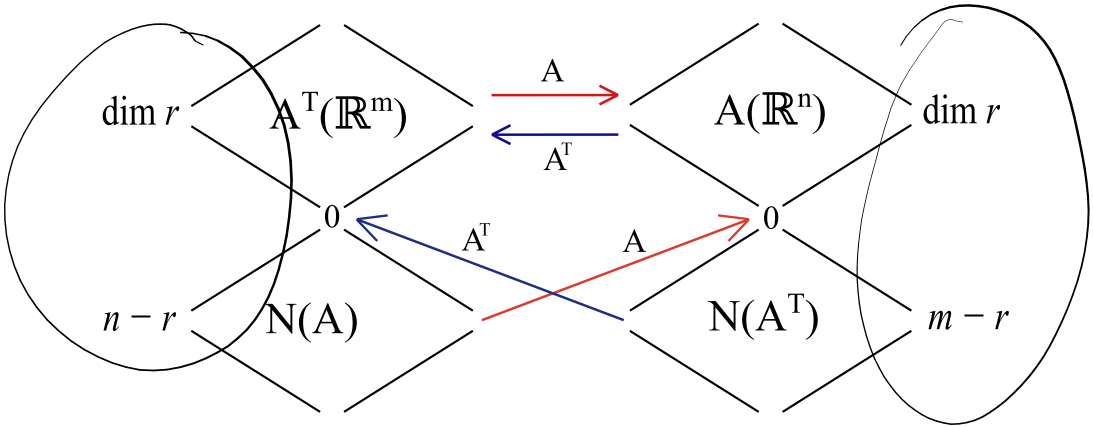
$$(I - A^*)x = f \text{ exists}$$

$$\Updownarrow$$

$$\langle f, f_i \rangle \quad \forall i$$

Fundamental Theorem of Linear Algebra





6.5 A Tiny Bit of Spectral Theory

Spectral Theory: Terminology

$A : X \to X$ bounded, λ is a ____ value:

Definition 9 (Eigenvalue) *There exists an element $\phi \in X$, $\phi \neq 0$ with $A\phi = \lambda\phi$.*

Definition 10 (Regular value) *The "resolvent" $(\lambda I - A)^{-1}$ exists and is bounded.*

Can a value be regular and "eigen" at the same time? No

What's special about ∞-dim here? → possible for λ to be neither eig val or regular val

Definition 11 (Resolvent set) $\rho(A) := \{\lambda \text{\textit{is regular}}\}$

Definition 12 (Spectrum) $\sigma(A) := \mathbb{C} \setminus \rho(A)$

$\sigma(A)$ includes eigenvalues,

but may include other

spectral values

Spectral Theory of Compact Operators

Theorem 10 *$A : X \to X$ compact linear operator, X ∞-dim.*
Then:

- $0 \in \sigma(A)$ *(show!)* → Suppose not. Then $(0-A)^{-1}$ exists → A^{-1} exists & is bounded → $AA^{-1} = I$ is compact → contradiction
- $\sigma(A) \setminus \{0\}$ *consists* only *of eigenvalues*
- $\sigma(A) \setminus \{0\}$ *is at most countable*
- $\sigma(A)$ *has no accumulation point except for 0*

Show first part.

Show second part. → Fredholm alternative

Rephrase last two: how many eigenvalues with $|\cdot| \geq R$?

$N(\lambda - A) = \{0\}$ or $\exists f \in N(\lambda - A)$ and $(\lambda - A)$ not bijection

Recap: What do compact operators do to high-frequency data?

Don't confuse $I - A$ with A itself!

$\sigma(A)$

dampen high frequency data

The high frequency eigenmodes are dampened

7 Singular Integrals and Potential Theory

Recap: Layer potentials

$G(x,y)$ – Green's function

Γ smooth

$-\frac{1}{2\pi} \log |x-y|$

$$(S\sigma)(x) := \int_\Gamma G(x-y)\sigma(y)ds_y$$

x deriv

$$(S'\sigma)(x) := PV\ \hat{n} \cdot \nabla_x \int_\Gamma G(x-y)\sigma(y)ds_y$$

$$(D\sigma)(x) := PV \int_\Gamma \hat{n} \cdot \nabla_y G(x-y)\sigma(y)ds_y$$

$$(D'\sigma)(x) := f.p.\ \hat{n} \cdot \nabla_x \int_\Gamma \hat{n} \cdot \nabla_y G(x-y)\sigma(y)ds_y$$

finite part

Definition 13 (Harmonic function) $\triangle u = 0$

Where are layer potentials harmonic?

on $\mathbb{R}^2 \setminus \Gamma$

by differentiation under the integral

On the double layer again

Is the double layer *actually* weakly singular?

Recap:

Definition 14 (Weakly singular kernel)

- *K defined, continuous everywhere except at $x = y$*
- *There exist $C > 0$, $\alpha \in (0, n-1]$ such that*

$$|K(x, y)| \leq C|x - y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

$K : \partial\Omega \times \partial\Omega \to \mathbb{R}$

$n = 2$

$\alpha \in (0, 1]$

$$|K(x,y)| \leq C|x-y|^{\alpha-n+1}$$
$$= C|x-y|^{\alpha-1}$$

$(-1, 0]$

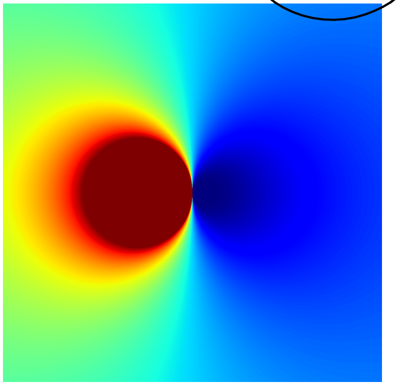
exponents of singularity

in 1D $\frac{1}{r^{.9999}}$ is integrable, but $\frac{1}{r}$ is not

$$\frac{\partial}{\partial_x}\log(|0-x|) = \frac{x}{x^2+y^2}$$

let $x=0$ or $y=0$

singular



$y=0$

- Singularity with approach on $y = 0$? nonsingular
- Singularity with approach on $x = 0$?

So life is simultaneously worse and better than discussed.

How about 3D? $(-x/|x|^3)$

Would like an analytical tool that requires ‘less’ fanciness.

Cauchy Principal Value

But I don't **want** to integrate across a singularity!

$$\int_{-1}^{1} \frac{1}{x} dx?$$

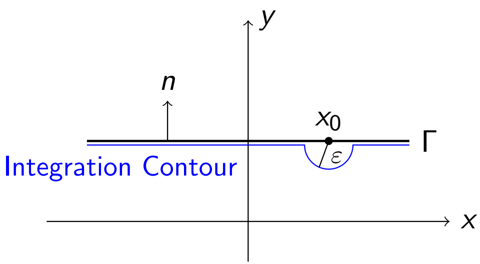
$$PV \int \frac{1}{x} dx = \lim_{\varepsilon \to 0} \left(\int_{-1}^{-\varepsilon} \frac{1}{x} dx + \int_{\varepsilon}^{1} \frac{1}{x} dx \right)$$

$$= 0$$

Depends on symmetry

$$\lim_{\varepsilon \to 0} \left(\int_{-1}^{-2\varepsilon} \frac{1}{x} dx + \int_{\varepsilon}^{1} \frac{1}{x} dx \right) \neq 0$$

Principal Value in n dimensions



Again: Symmetry matters!

What about even worse singularities? "finite part"

$$\mathrm{PV} \int_G f(x)\,dx = \lim_{\varepsilon \to 0} \int_{G \setminus B_\varepsilon(x)} f(x)\,dx$$

Recap: Layer potentials

$$(S\sigma)(x) := \int_\Gamma G(x-y)\sigma(y)ds_y$$
$$(S'\sigma)(x) := \text{PV }\hat{n}\cdot\nabla_x\int_\Gamma G(x-y)\sigma(y)ds_y$$
$$(D\sigma)(x) := \text{PV }\int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\sigma(y)ds_y$$
$$(D'\sigma)(x) := f.p.\ \hat{n}\cdot\nabla_x\int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\sigma(y)ds_y$$

Important for us: Recover 'average' of interior and exterior limit without having to refer to off-surface values.

Green's Theorem

Theorem 11 (Green's Theorem [Kress LIE Thm 6.3])

$$(i) \quad \int_D u\triangle v + \nabla u \cdot \nabla v = \int_{\partial D} u(\hat{n} \cdot \nabla v)ds$$

$$(ii) \quad \int_D u\triangle v - v\triangle u = \int_{\partial D} u(\hat{n} \cdot \nabla v) - v(\hat{n} \cdot \nabla u)ds$$

If $\triangle v = 0$, then $\quad \nabla u = 0$

by (i) let $u = 1$ $\quad \Delta v = 0 \quad \int_{\partial D} \hat{n} \cdot \nabla v =? = 0$

What if $\triangle v = 0$ and $u = G(|y - x|)$ in Green's second identity?

(ii) $\quad \int -v\Delta u = v(x) = S[\partial_n v] - D[v]$

$$v(x) = \int_{\partial D} G(x,y)\,(n \cdot \nabla v) - \int_{\partial D} v \; n \cdot \nabla G(x,y)$$

Green's Formula

Theorem 12 (Green's Formula **[Kress LIE Thm 6.5])** *If $\triangle u = 0$, then*

$$(S(\hat{n}\cdot\nabla u) - Du)(x) = \begin{cases} u(x) & x \in D \\ \frac{u(x)}{2} & x \in \partial D \\ 0 & x \notin D \end{cases}$$

Suppose I know 'Cauchy data' $(u|_{\partial D}, \hat{n}\cdot\nabla u|_{\partial D})$ of u. What can I do?

What if D is an exterior domain?

no longer holds

Can recover u inside D.