TODAY

hw3 graded

hw5 → Dec 4

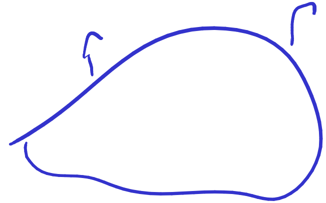
Last time:

- Green's formula int/ext
- jump relations
- BVP uniqueness

next:

- IE uniqueness
- Helmholtz
- Calderón identities
- numerics

$u(x) = \int_\Gamma \hat{n}_y \cdot \nabla_y G(x-y)\,\sigma(y)\,dy$



Boundary Value Problems: Overview

	Dirichlet	**Neumann**
Int.	$\lim_{x\to\partial\Omega-} u(x) = g$ ⊕ unique MP	$\lim_{x\to\partial\Omega-} \hat{n}\cdot\nabla u(x) = g$ ◐ may differ by constant
Ext.	$\lim_{x\to\partial\Omega+} u(x) = g$ $u(x) = \begin{cases} O(1) & 2D \\ o(1) & 3D \end{cases}$ as $\lvert x\rvert \to \infty$ ⊕ unique	$\lim_{x\to\partial\Omega+} \hat{n}\cdot\nabla u(x) = g$ $u(x) = o(1)$ as $\lvert x\rvert \to \infty$ ⊕ unique

$\int \lvert\nabla(u_1 - u_2)\rvert^2 = 0$

$\nabla G(\vec{x}) = \frac{\vec{x}}{\lvert\vec{x}\rvert^2}$

with $g \in C(\partial\Omega)$.

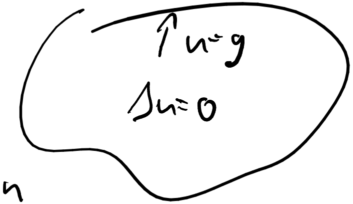
What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)

Dirichlet uniqueness: why?

Neumann uniqueness: why?

Truth in advertising: Missing assumptions on Ω?

What's a DtN map?



Next mission: Find IE representations for each.

	Dirichlet	Neumann
int	repr. $u(x) = D\sigma(x)$ IE $\left(\left(\frac{I}{2} - D\right)\sigma\right)(x) = g(x)$	$u(x) = S\sigma(x)$ IE: $\left(\frac{I}{2} + S'\right)\sigma(x)$
ext.	repr. $u(x) = D\sigma(x)$ IE $\frac{I}{2} + D$	repr $u(x) = S\sigma(x)$ IE $\frac{I}{2} - S'$

$S \qquad D$

$S' \qquad D'$ — not compact; bounded

Uniqueness of Integral Equation Solutions

int. dir

Theorem 17 (Nullspaces [Kress LIE 2nd ed. Thm 6.20]**)**

- $N(I/2 - D) = N(I/2 - S') = \{0\}$ ext. Neumann
- $N(I/2 + D) = \operatorname{span}\{1\}$, $N(I/2 + S') = \operatorname{span}\{\psi\}$, *where* $\int \psi \neq 0$.

int. Neumann

FH! $\dim N(I - A) = \dim N(I - A^*)$

Show $N(I/2 - D) = \{0\}$. ✓

Show $N(I/2 - S') = \{0\}$. ✓ (FH)

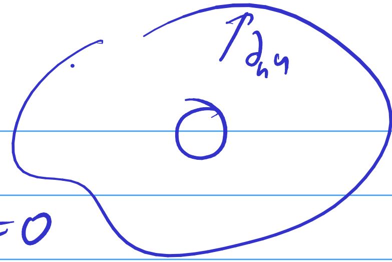
Show $N(I/2 + D) \supseteq \operatorname{span}\{1\}$.

What extra conditions on the RHS do we obtain?

$(I - A)(X) = N(I - A^*)^{\perp}$

$\Rightarrow$ range of int. Neumann $\perp_{L^2}$ const.

$[S\varphi] =$



Show $N\left(\frac{I}{2} - D\right) = \{0\}$.

Suppose: $\frac{\varphi}{2} - D\varphi = 0$. To show: $\varphi = 0$

$u(x) := D\varphi(x)$. $\Delta u(x) = 0$ off of Γ.

$\lim_{x \to \Gamma^-} u^- = D\varphi - \varphi/2 = 0$

$u|_{int} = 0$ because of int BVP uniqueness

$(\partial_n u)_- = 0 = (\partial_n u)_+$

$\Rightarrow u|_{ext} = 0$ By jump cond. $\varphi = u_+ - u_- = 0$

$\rightarrow$ "Clean" Existence for 3 out of 4.

Patching up Exterior Dirichlet (skipped)

Problem: $N(I/2 + S') = \{\psi\}$...but we do not know ψ.

Use a different kernel:

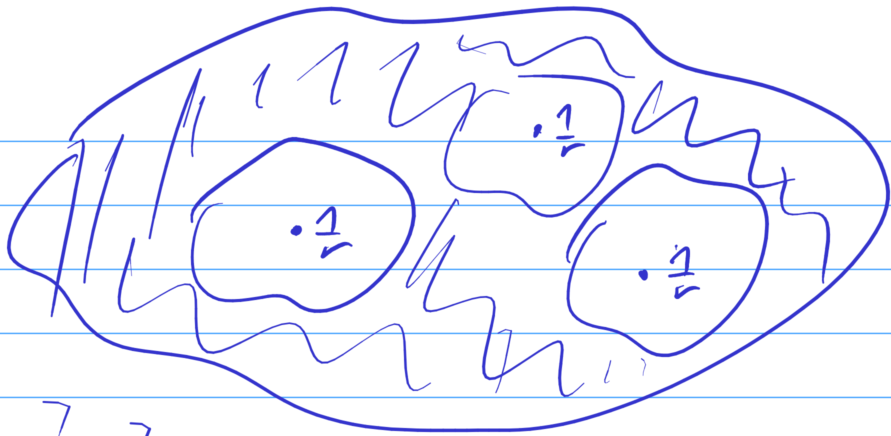
$$\hat{n} \cdot \nabla_y G(x, y) \quad \to \quad \int \left(\hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}} \right) \sigma(y)\, dy$$

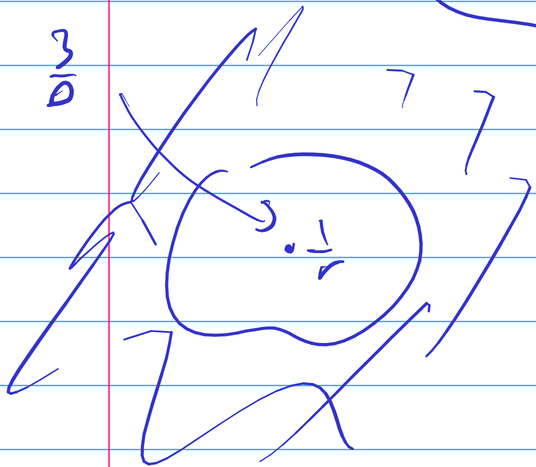
Note: Singularity only at origin! (assumed $\in \Omega$)

$$D\sigma(x) + \frac{1}{|x|^{n-2}} \int \sigma(y)\, dy$$

- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce $u = 0$ on exterior.
- $|x|^{n-2} u(x) =?$ on exterior
- Thus $\int \phi = 0$. Contribution of the second term?
- $\phi/2 + D\phi = 0$, i.e. $\phi \in N(I/2 + D) =?$

$\frac{3}{0}$





"bad"
Helmholtz

$$\Delta u = 0 \quad \rightsquigarrow \quad \Delta u + k^2 u = 0$$

$$\Delta u - k^2 u = 0$$

"good"

$u = e^{i\omega t} \tilde{u}$

Yukawa
Poisson-Boltzmann

$\partial_t^2 u = \Delta u$

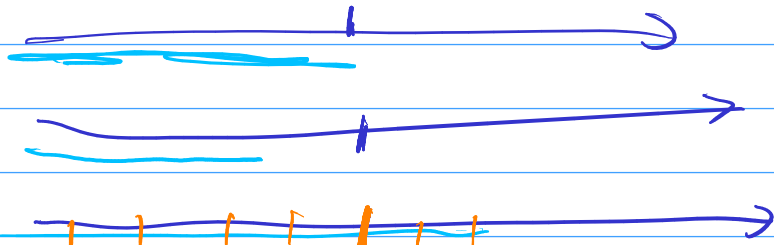
$(i\omega)^2$

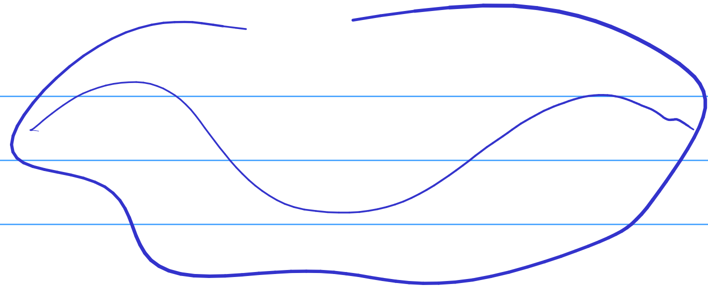
$-\Delta u$ has nonneg eigenvalues

Δu:

(neg. definite) $\Delta u - k^2$

$\Delta u + k^2 u$





$$\Delta u + k^2 u = 0$$

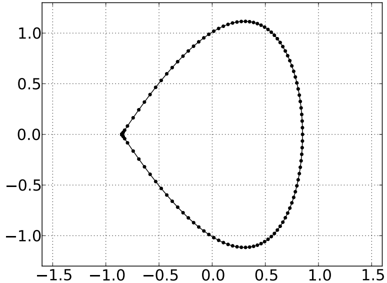
$$u = 0$$

- Existence/uniqueness?

$\rightarrow$ Existence for 4 out of 4.

Remaining key shortcoming of IE theory for BVPs?

Domains with Corners



What's the problem? *(Hint: Jump condition for constant density)*

At corner x_0: (2D)

$$\lim_{x \to x_0 \pm} = \int_{\partial\Omega} \hat{n} \cdot \nabla_y G(x, y) \phi(y) ds_y \pm \frac{1}{2} \frac{\langle \text{opening angle on } \pm \text{ side} \rangle}{\pi} \phi$$

$\rightarrow$ non-continuous behavior of potential on Γ at x_0

What space have we been living in?

Fixes:

- I + Bounded (Neumann) + Compact (Fredholm)

- Use L^2 theory

  (point behavior "invisible")

Numerically: Needs consideration, but ultimately easy to fix.

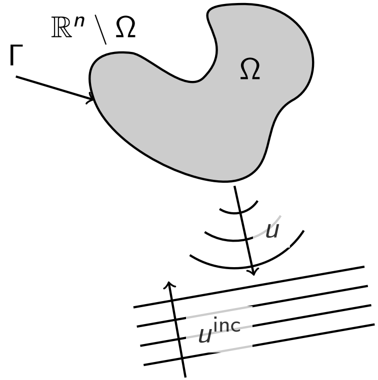
8.2 Helmholtz

Where does Helmholtz come from?

Derive the Helmholtz equation from the wave equation

$$\partial_t^2 U = c^2 \triangle U,$$

The prototypical Helmholtz BVP: A Scattering Problem



Ansatz:

$$u^{\mathrm{tot}} = u + u^{\mathrm{inc}}$$

Solve for scattered field u.