TODAY:

- Nyström ←
- Quadrature ←

$A_n \to A$

A_n:

$$\begin{pmatrix} K_1 & K_2 \\ K_2 & K_1 \end{pmatrix}$$

Δ_Γ

$u = G * \varphi$

$A_n : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\wr$

$C(\Gamma) \to C(\Gamma)$

$A : C(\Gamma) \to C(\Gamma)$

$\|A_n - A\| \overset{?}{\to}$ ✓

$1 - A_n = \epsilon$

$\varphi = A_n\varphi + f$

$$\int G(x,y)\,\sigma(y)\,dy$$

$$\sum \omega_i G(x, y_i)\,\sigma(y_i)$$

$\Rightarrow \|A_n - A\|$ ✗ won't work

$\|(A_n - A) A\|$ ✓

Compactness-Based Convergence

X Banach space (think: of functions)

Theorem 19 (Not-quite-norm convergence **[Kress LIE 2nd ed. Cor 10.4])** $A_n : X \to X$ *bounded linear operators,*
functionwise convergent to $A : X \to X$
Then convergence is uniform on compact subsets $U \subset X$, *i.e.*

$$\sup_{\phi \in U} \|A_n\phi - A\phi\| \to 0 \qquad (n \to \infty)$$

(uniform boundedness principle)

How is this different from norm convergence?

Set $\mathcal{A}$ of operators $A : X \to X$

$\{A_n : n \in \mathbb{N}\}$

Definition 15 (Collectively compact) $\mathcal{A}$ *is called* collectively compact *if and only if*
for $U \subset X$ *bounded,* $\mathcal{A}(U)$ *is relatively compact.*

What was relative compactness (=precompactness)?

Is each operator in the set $\mathcal{A}$ compact?

When is a sequence collectively compact?

Is the limit operator of such a sequence compact?

How can we use the two together?

Making use of Collective Compactness

X Banach space, $A_n : X \to X$, (A_n) collectively compact, $A_n \to A$ functionwise.

Corollary 1 (Post-compact convergence [Kress LIE 2nd ed. Cor 10.8])

- $\|(A_n - A)A\| \to 0$
- $\|(A_n - A)A_n\| \to 0$

$(n \to \infty)$

Anselone's Theorem

Assume:

$(I - A)\varphi = f$

$(I-A)^{-1}$ exists, with $A : X \to X$ compact, $(A_n) : X \to X$ collectively compact and $A_n \to A$ functionwise.

Theorem 20 (Nyström error estimate [Kress LIE 2nd ed. Thm 10.9]) *For sufficiently large n, $(I - A_n)$ is invertible and*

$$\|\phi_n - \phi\| \leq C(\|(A_n - A)\phi\| + \|f_n - f\|)$$

$(A_n - A)\phi$ → quadrature error

$$C = \frac{1 + \|(I-A)^{-1}A_n\|}{1 - \|(I-A)^{-1}(A_n - A)A_n\|}$$

$I + (I-A)^{-1}A_n = ?\ (I - A_n)^{-1}$

$1 + \frac{a}{1-a} = \frac{1-a}{1-a} + \frac{a}{1-a} = \frac{1}{1-a}$

Show the theorem.

Nyström: *specific to I+compact.* Why?

$$B_n = I + (I-A)^{-1} A_n$$

$$I \leftarrow B_n (I - A_n) = I - S_n$$

$$S_n = (I-A)^{-1} \underbrace{(A - A_n)}_{\|\cdot\| \to 0} A_n$$

$$(I - S_n)^{-1} = \sum_{i=0}^{\infty} (S_n)^i$$ assuming $\|S_n\| < 1$ (pick n large enough)

$$(I - A_n)^{-1} = (I - S_n)^{-1} B_n$$

Nyström: Collective Compactness

$\int f \approx \sum_{i=0}^{n} w_i f(x_i)$ ← yields A_n

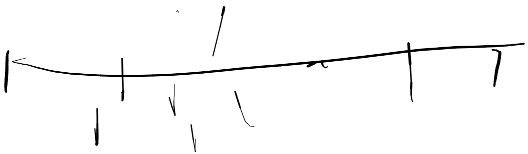
Assume

$$\sum |\text{quad. weights for } n \text{ points}| \leq C \qquad \text{(independent of } n\text{)} \qquad (3)$$

We've *assumed* collective compactness. Do we have that?

Also assumed functionwise uniform convergence, i.e. $\|A_n\phi - A\phi\| \to 0$ for each ϕ.

Arzela-Ascoli: need bounded and equicont. in range space of n



oscillatory weights

$\sum w_i = b-a$

$\|A_n \varphi\| = \|\sum k(x, y_i) w_i \varphi(y_i)\|$

$\leq \sum |w_i| \max_y |K(x, y_i)\varphi(y)|$

9.3 Integral Equation Discretizations: Projection

Error Estimates for Projection

X Banach spaces, $A : X \to X$ injective, $P_n : X \to X_n$

Theorem 21 (Céa's Lemma [Kress LIE 2nd ed. Thm 13.6]) *Convergence of the projection method*
$\Leftrightarrow$ *There exist n_0 and M such that for $n \geq n_0$*

1. *$P_n A : X_n \to X_n$ are invertible,*
2. *$\|(P_n A)^{-1} P_n A\| \leq M$.*

In this case,

$$\|\phi_n - \phi\| \leq (1 + M) \inf_{\psi \in X_n} \|\phi - \psi\|$$

Proof? (skipped)

Core message of the theorem?

What goes into P_n?

Note domain of invertibility for $P_n A$.

Domain/range of $(P_n A)^{-1} P_n A$?

Relationship to conditioning?

Relationship to second-kind?

Exact projection methods: hard. (Why?) What if we implement a perturbation? (i.e. apply quadature instead of computing exact integrals?)

Decisions, Decisions: Nyström or Galerkin?

Quote Kress LIE, 2nd ed., p. 244 (Sec. 14.1):

> [...] the Nyström method is generically stable whereas the collocation and Galerkin methods may suffer from instabilities due to a poor choice of basis for the approximating subspace.

Quote Kress LIE, 2nd ed., p. 244 (Sec. 13.5):

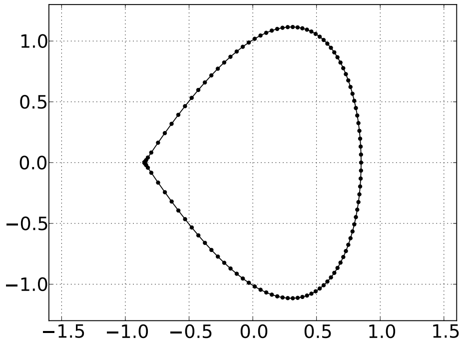
> In principle, for the Galerkin method for equations of the second kind the same remarks as for the collocation method apply. As long as numerical quadratures are available, in general, the Galerkin method cannot compete in efficiency with the Nyström method.
> Compared with the collocation method, it is less efficient, since its matrix elements require double integrations.

Need good quadratures to use Nyström.

Remaining advantage of Galerkin:

Can be made not to break for non-second-kind.

Galerkin without the Pain [Bremer et al. '11]



Problem: Singular behavior at corner points. Density may blow up.

Can the density be convergent in the $\|\cdot\|_\infty$ sense?

Conditioning of the discrete system?

GMRES will flail and break, because it sees $\ell^2 \sim l^\infty \sim L^\infty$ convergence.

Make GMRES 'see' L^2 convergence by redefining density DOFs:

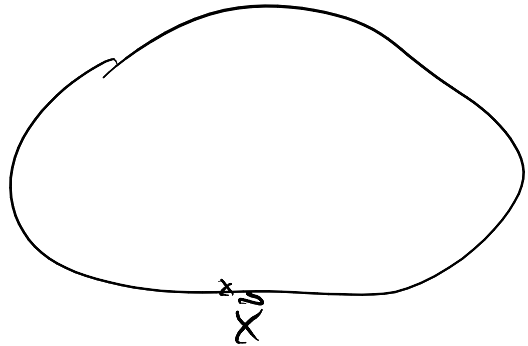
$$\overline{\boldsymbol{\sigma}}_h := \begin{pmatrix} \sqrt{\omega_1}\sigma(x_1) \\ \vdots \\ \sqrt{\omega_n}\sigma(x_n) \end{pmatrix} = \sqrt{\boldsymbol{\omega}}\boldsymbol{\sigma}_h$$

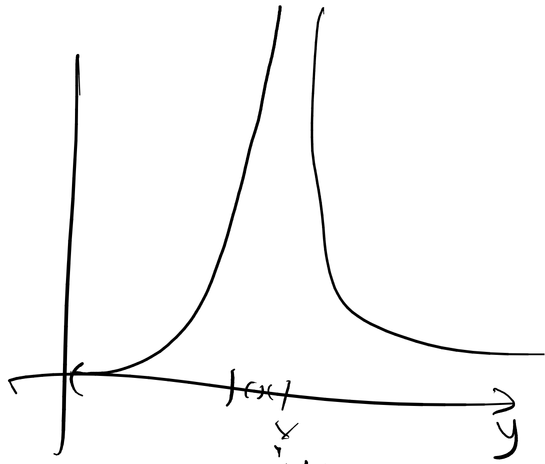
So $\overline{\boldsymbol{\sigma}}_h \cdot \overline{\boldsymbol{\sigma}}_h =?$

Also fixes system conditioning! Why?

10 Computing Integrals: Approaches to Quadrature

$$\int_\Gamma \log(|x-y|)\,\sigma(y)\,dy$$





Error est. for Gaussian quad:

$$\left|Q_n(f) - \int_a^b f\right| \le C \cdot \|f^{(2q)}\|_\infty h^{2q}$$

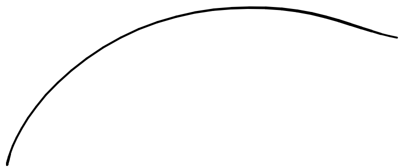
- adaptive quad (if integrable)

‘Off-the-shelf’ ways to compute integrals

How do I compute an integral of a nasty singular kernel?

Symbolic integration

Why not Gaussian?

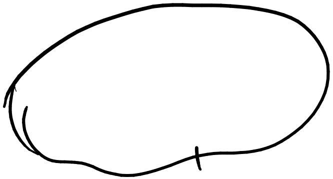


Kussmaul-Martensen quadrature

Theorem 22 (A special integral [Kress LIE Lemma 8.21]**)**

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left(4\sin^2\frac{t}{2}\right) e^{imt}\,dt = \begin{cases} 0 & m = 0, \\ -\frac{1}{|m|} & m = \pm 1, \pm 2 \ldots . \end{cases}$$

Why is that exciting?



$\gamma : [0, 2\pi] \to \mathbb{R}^2$

$$\int_0^{2\pi} G(x, \gamma(t))\, \sigma(t)\, |\gamma'(t)|\, dt$$

$$\sigma(t)\,|\gamma'(t)| \approx \sum_{m=-\ell}^{\ell} \alpha_m e^{imt}$$

$$= \sum_m \alpha_m \int \log(\sin^2)\, e^{imt} + \int (\log(\sin^2) - G)$$

Singularity Subtraction

$$\int \langle \text{Thing } X \text{ you would like to integrate} \rangle$$
$$= \int \langle \text{Thing } Y \text{ you } \textit{can} \text{ integrate} \rangle$$
$$+ \int \langle \text{Difference } X - Y \text{ which is easy to integrate (numerically)} \rangle$$

Give a typical application.

Drawbacks?

High-Order Corrected Trapezoidal Quadrature

- Conditions for new nodes, weights

  ($\rightarrow$ linear algebraic system, dep. on n)

  to integrate

  $$\langle\text{smooth}\rangle \cdot \langle\text{singular}\rangle + \langle\text{smooth}\rangle$$

- Allowed singularities: $|x|^{\lambda}$ (for $|\lambda| < 1$), $\log|x|$
- Generic nodes and weights for log singularity
- Nodes and weights copy-and-pasteable from paper

[Kapur, Rokhlin '97]

Alpert '99 conceptually similar:

Generalized Gaussian

- “Gaussian”:
  - Integrates $2n$ functions exactly with n nodes
  - Positive weights
- Clarify assumptions on system of functions (“Chebyshev system”) for which Gaussian quadratures exist
- When do (left/right) singular vectors of integral operators give rise to Chebyshev systems?
  - In many practical cases!
- Find nodes/weights by Newton’s method
  - With special starting point
- Very accurate
- Nodes and weights for download

[Yarvin/Rokhlin ‘98]

Singularity cancellation: Polar coordinate transform

$$\int \int_{\partial\Omega} K(\mathbf{x},\mathbf{y})\phi(y)ds_y$$

$$=$$

$$\int_0^R \int_{\mathbf{x}+\mathbf{r}\in\partial\Omega\cap\partial B(\mathbf{x},r)} K(\mathbf{x},\mathbf{x}+\mathbf{r})\phi(\mathbf{x}+\mathbf{r})d\langle\text{angles}\rangle\, r\, dr$$

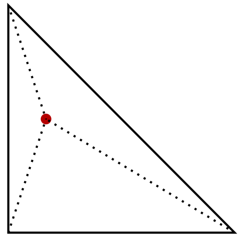
$$=$$

$$\int_0^R \int_{\mathbf{x}+\mathbf{r}\in\partial\Omega\cap\partial B(\mathbf{x},r)} \frac{K_{\text{less singular}}(\mathbf{x},\mathbf{x}+\mathbf{r})}{r}\phi(\mathbf{x}+\mathbf{r})d\langle\text{angles}\rangle\, r\, dr$$

where $K_{\text{less singular}} = K \cdot r$.

PCT

Quadrature on triangles



Problem: Singularity can sit *anywhere* in triangle

$\rightarrow$ need *lots* of quadrature rules (one per target)

Kernel regularization

Singularity makes integration troublesome: *Get rid of it!*

$\varepsilon = \frac{1}{2}$ $\quad$ $\varepsilon = \frac{1}{4}$ $\quad$ $\frac{1}{8}$

$$\frac{\cdots}{\sqrt{(x-y)^2}} \quad \to \quad \frac{\cdots}{\sqrt{(x-y)^2+\epsilon^2}}$$

Use Richardson extrapolation to recover limit as $\epsilon \to 0$.

(May also use geometric motivation: limit along line towards singular point.)

Primary drawbacks:

- Low-order accurate
- Need to make ϵ smaller (i.e. kernel more singular) to get better accuracy

Can take many forms–for example:

- Convolve integrand to smooth it

  ($\to$ remove/weaken singularity)
- Extrapolate towards no smoothing

Related: [Beale/Lai '01]

10.1 Quadrature by expansion ('QBX')

(see the corresponding section of `http://bit.ly/1Msw0EQ`)

11 Going General: More PDEs