TODAY:

- Obt Wi

- Helmholtz: The bad paper

- Calderón

- numeric
What's the problem? \( \text{(Hint: Jump condition for constant density)} \)

At corner \( x_0 \): (2D)

\[
\lim_{x \to x_0} = \int_{\partial \Omega} \hat{n} \cdot \nabla_y G(x, y) \phi(y) ds_y \pm \frac{1}{2} \left( \text{opening angle on } \pm \text{ side} \right) \frac{\phi}{\pi}
\]

\( \rightarrow \) non-continuous behavior of potential on \( \Gamma \) at \( x_0 \)

What space have we been living in?

Fixes:

- \( I + \text{Bounded (Neumann)} + \text{Compact (Fredholm)} \)
• Use $L^2$ theory
  
  (point behavior “invisible”)

Numerically: Needs consideration, but ultimately easy to fix.
8.2 Helmholtz
Where does Helmholtz come from?

Derive the Helmholtz equation from the wave equation

$$\partial_t^2 U = c^2 \triangle U,$$
The prototypical Helmholtz BVP: A Scattering Problem

Ansatz:

$$u^{\text{tot}} = u + u^{\text{inc}}$$

Solve for scattered field $u$. 
Helmholtz: Some Physics

Physical quantities:

- Velocity potential: $U(x, t) = u(x)e^{-i\omega t}$
  (fix phase by e.g. taking real part)
- Velocity: $v = (1/\rho_0)\nabla U$
- Pressure: $p = -\partial_t U = i\omega u e^{-i\omega t}$
  - Equation of state: $p = f(\rho)$

What's $\rho_0$?

What happens to a pressure BC as $\omega \to 0$?
Helmholtz: Boundary Conditions

- **Sound-soft**: Pressure remains constant
  - Scatterer “gives”
  - \( u = f \rightarrow \text{Dirichlet} \)

- **Sound-hard**: Pressure same on both sides of interface
  - Scatterer “does not give”
  - \( \hat{n} \cdot \nabla u = 0 \rightarrow \text{Neumann} \)

- **Impedance**: Some pressure translates into motion
  - Scatterer “resists”
  - \( \hat{n} \cdot \nabla u + ik \lambda u = 0 \rightarrow \text{Robin} \ (\lambda > 0) \)

- **Sommerfeld** radiation condition: allow only outgoing waves
  \[
  r^{\frac{n-1}{2}} \left( \frac{\partial}{\partial r} - ik \right) u(x) \rightarrow 0 \quad (r \rightarrow \infty)
  \]
  (where \( n \) is the number of space dimensions)
Many interesting BCs → many IEs! :) 

Transmission between media: What’s continuous?
Unchanged from Laplace

**Theorem 18 (Green’s Formula [Colton/Kress IAEST Thm 2.1])** If \( \Delta u + k^2 u = 0 \), then

\[
(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} 
  u(x) & x \in D \\
  \frac{u(x)}{2} & x \in \partial D \\
  0 & x \notin D
\end{cases}
\]

\[
\lim_{x \to x_0 \pm} (S'u) = \left( S' \mp \frac{1}{2}I \right)(u)(x_0) \quad \Rightarrow \quad [S'u] = -u
\]

\[
\lim_{x \to x_0 \pm} (Du) = \left( D \pm \frac{1}{2}I \right)(u)(x_0) \quad \Rightarrow \quad [Du] = u = 0
\]

Why is singular behavior (esp. jump conditions) unchanged?

\[
e^{i\kappa r} - \frac{1}{r} = \frac{1}{r} + \frac{e^{i\kappa r} - 1}{r}
\]
Why does Green’s formula survive?
Remember Green’s theorem:
\[
\int_{\Omega} u \Delta v - v \Delta u = \int_{\partial \Omega} u(\hat{n} \cdot \nabla v) - v(\hat{n} \cdot \nabla u) \, ds
\]
Resonances

$-\Delta$ on a bounded (interior) domain with homogeneous Dirichlet/Neumann BCs has countably many real, positive eigenvalues.

What does that have to with Helmholtz?

Why could it cause grief?
Helmholtz: Boundary Value Problems

Find $u \in C(\bar{D})$ with $\triangle u + k^2 = 0$ such that

<table>
<thead>
<tr>
<th>Dirichlet</th>
<th>Neumann</th>
</tr>
</thead>
<tbody>
<tr>
<td>Int.</td>
<td>Neumann</td>
</tr>
<tr>
<td>[ \lim_{x \to \partial D^-} u(x) = g ] unique (−resonances)</td>
<td>[ \lim_{x \to \partial D^-} \hat{n} \cdot \nabla u(x) = g ] unique (−resonances)</td>
</tr>
<tr>
<td>Ext.</td>
<td>Neumann</td>
</tr>
<tr>
<td>[ \lim_{x \to \partial D^+} u(x) = g ] Sommerfeld unique</td>
<td>[ \lim_{x \to \partial D^+} \hat{n} \cdot \nabla u(x) = g ] Sommerfeld unique</td>
</tr>
</tbody>
</table>

with $g \in C(\partial D)$.

Find layer potential representations for each.

\[
\eta(x) = S\sigma - i\mathcal{O}(\sigma)
\]

"CFIE"

Combined-field ext Neumann: \[
(\frac{1}{2} - S' - \mathcal{O}')(\sigma) = g
\]
Patching up resonances

**Issue:** Ext. IE inherits non-uniqueness from ‘adjoint’ int. BVP

**Fix:** Tweak representation [Brakhage/Werner ’65, ...]

(also called the ‘CFIE’–‘combined field integral equation’)

\[ u = D\phi - i\alpha S\phi \]

(\(\alpha\): tuning knob \(\rightarrow 1\) is fine, \(\sim k\) better for large \(k\))

How does this help?

Uniqueness for remaining IEs similar. (skipped)
8.3 Calderón identities

Show that $D'$ is self adjoint.

Show that $(S\varphi, D'\psi) = ((S' + 1/2)\varphi, (D - 1/2)\psi)$.

$(\varphi, SD'\psi)$?

\[
\begin{align*}
(D'\varphi, \psi) &= (\varphi, D'\psi) \\
\text{with } u &= D\varphi \quad \text{and } v = D\psi \\
\Delta u &= \text{div}(\nu \Delta \eta) = 0 \quad \text{on } \partial \Omega \quad \text{and} \\
\text{div}(\nu \Delta \eta) &= 0.
\end{align*}
\]
\[ S \theta_{1/2} = (\hat{0} + \theta(z), C_\theta) = (0, 0) \]

\[ (\theta_0, \theta_1) = (\chi, \chi) \]

\[ (\theta_0 + \theta_1, \theta_0 - \theta_1) = (0, 0) \]
Calderón Identities: Summary

- $SD' = D^2 - 1/4$
- $D'S = S^2 - 1/4$

Also valid for Laplace (jump relation same after all!)

Why do we care?
9 Back from Infinity: Discretization

9.1 Fundamentals: Meshes, Functions, and Approximation
Numerics: What do we need?

- Discretize curves and surfaces
  - Interpolation
  - Grid management
  - Adaptivity
- Discretize densities
- Discretize integral equations
  - Nyström, Collocation, Galerkin
- Compute integrals on them
  - “Smooth” quadrature
  - Singular quadrature
- Solve linear systems
Constructing Discrete Function Spaces

Floating point numbers \( \rightarrow \) Functions

(Degrees of Freedom/DoFs)

Discretization relies on three things:

- Base/reference domain
- Basis of functions
- Meaning of DoFs

Related finite element concept: Ciarlet triple

Discretization options for a curve?
What do the DoFs mean?

Common DoF choices:

- Point values of function
- Point values of (directional?) derivatives
- Basis coefficients
- Moments

Often: useful to have both “modes”, “nodes”, jump back and forth
Why high order?

Order $p$: Error bounded as

$$|u_h - u| \leq C h^p$$

Thought experiment:

<table>
<thead>
<tr>
<th>Order</th>
<th>1,000 DoFs</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>First order</td>
<td>$\approx 1,000$ triangles</td>
<td>0.1</td>
</tr>
<tr>
<td>Fifth order</td>
<td>$\approx 66$ triangles</td>
<td>0.1</td>
</tr>
<tr>
<td>Error: 0.01</td>
<td>$\rightarrow ?$</td>
<td></td>
</tr>
</tbody>
</table>

Remarks:

- Want $p \geq 3$ available.
- **Assumption:** Solution sufficiently smooth.
- Ideally: $p$ chosen by user
What is an Unstructured Mesh?

Why have an unstructured mesh?

What is the trade-off in going unstructured?
# Fixed-order vs Spectral

<table>
<thead>
<tr>
<th>Fixed-order</th>
<th>Spectral</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of DoFs $n$</td>
<td>Number of DoFs $n$</td>
</tr>
<tr>
<td>$\sim$</td>
<td>$\sim$</td>
</tr>
<tr>
<td>Number of ‘elements’</td>
<td>Number of modes resolved</td>
</tr>
<tr>
<td>Error $\sim \frac{1}{n^p}$</td>
<td>Error $\sim \frac{1}{C^n}$</td>
</tr>
</tbody>
</table>

Examples?
- Piecewise Polynomials
- Global Fourier
- Global Orth. Polynomials

What assumptions are buried in each of these?

What should the DoFs be?
What’s the difficulty with purely modal discretizations?
Vandermonde Matrices

\[
\begin{pmatrix}
    x_0^0 & x_0^1 & \cdots & x_0^n \\
    x_1^0 & x_1^1 & \cdots & x_1^n \\
    \vdots & \vdots & \ddots & \vdots \\
    x_n^0 & x_n^1 & \cdots & x_n^n \\
\end{pmatrix}
\begin{pmatrix}
    a_0 \\
    a_1 \\
    \vdots \\
    a_n \\
\end{pmatrix}
= ?
\]
Generalized Vandermonde Matrices

\[
\begin{pmatrix}
\phi_0(x_0) & \phi_1(x_0) & \cdots & \phi_n(x_0) \\
\phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_n(x_1) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_0(x_n) & \phi_1(x_n) & \cdots & \phi_n(x_n)
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
\vdots \\
a_n
\end{pmatrix}
= ?
\]
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\vdots & \vdots & \ddots & \vdots \\
\phi_0(x_n) & \phi_1(x_n) & \cdots & \phi_n(x_n)
\end{pmatrix}
\]

MODAL COEFFS = NODAL COEFFS

Node placement?

Vandermonde conditioning?

What about multiple dimensions?
Common Operations

(Generalized) Vandermonde matrices simplify common operations:

- Modal ↔ Nodal ("Global interpolation")
  - Filtering
  - Up-/Oversampling
- Point interpolation (Hint: solve using $V^T$)
- Differentiation
- Indefinite Integration
- Inner product
- Definite integration