Today

\[ N \times N \quad k \leq N \]

\[ u = g \]

\[ N^\alpha k^\beta \]

\[ \Delta u = 0 \]

[Diagram: A rectangle labeled 'N', with a smaller rectangle inside it labeled 'N' and a line segment from 'N' to the bottom]

Right way: \[ N^2 k \]

Right way: \[ N k^2 \]

Right way: \[ RRQR \]

Right way: \[ \text{ID} \]
Sometimes the SVD is too good (aka expensive)—we may need less accuracy/weaker promises, for a significant decrease in cost.

\[ A \Pi = QR \]

\[ = Q \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \]

\[ \sigma_{w+1} \leq \|R_{i2}\|_2 \]
Interpolative Decomposition (ID)

Sometimes it would be helpful to know which columns of $A$ contribute ‘the most’ to the rank. (rather than have the waters muddied by an orthogonal transformation like in QR)

\[
A \approx A_{[; , J]} \rho \\
J \in \mathbb{N}^k
\]
Set \( B = Q \mathcal{N}_{11} \).

\[ B = \begin{pmatrix} \mathbf{1} & \mathbf{0} \end{pmatrix} \mathcal{N}_{11} \mathcal{N}_{11}^{-1} \mathcal{R}_{12} \begin{pmatrix} \mathbf{1} & \mathbf{0} \end{pmatrix} \]

\[ B P = Q \mathcal{N}_{11} \left( \begin{pmatrix} \mathbf{1} & \mathbf{0} \end{pmatrix} \mathcal{N}_{11}^{-1} \mathcal{R}_{12} \right) = Q \left( \begin{pmatrix} \mathbf{1} & \mathbf{0} \end{pmatrix} \mathcal{N}_{11} \mathcal{N}_{11}^{-1} \mathcal{R}_{12} \right) = A \]

\( S \mathbf{A} = Q \mathcal{A} (\mathcal{R}_{11}, \mathcal{R}_{12}) \quad \text{(is well-cond)} \)

- entries \( 1 \cdot 1 \leq 2 \).
A \cdot B
What does the ID buy us?

Specifically: Name a property that the ID has that other factorizations do not have.

\[ A \approx QQ^\top A \]

\[ Q = P\begin{bmatrix} Q_{(:,1)} \end{bmatrix} \]

\[ A_{(:,1)} = \begin{bmatrix} QQ^\top A \end{bmatrix}_{(:,1)} \]

\[ = \begin{bmatrix} P \begin{bmatrix} Q_{(:,1)} \end{bmatrix} \end{bmatrix} Q^\top A \]

\[ A = A_{(:,1)} P \]
\[
= \left( p_{C j} \circ d \right) Q_{C j} \circ Q^T A \\
= Q_{C j} \circ Q^T A \\
\approx A
\]
<table>
<thead>
<tr>
<th>ID</th>
<th>Q vs ID</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td><strong>What does row selection mean for the LRA?</strong></td>
</tr>
</tbody>
</table>
Leveraging the ID

Build a low-rank SVD with row extraction.

1. Obtain row subset \( J \) and up sampler \( P \).

2. Compute a row QR

\[
(A \{J; J\})^T = Q \bar{R}
\]

where \( N \times k \) and \( k \times k \). (Formula)

3. Upsample row coefficients: \( \bar{R} \)

\[
\bar{Z} = P \bar{R}^T
\]

4. SVD \( \bar{Z} = U \Sigma V^T \)

\[
N \text{ \footnotesize \{ k \}} \quad N^2 k \leq kn^2
\]
\[ u \in (\hat{Q} \hat{\nu})^r \]

\[ = \gamma \sqrt{Q} \]

\[ = \sqrt{P \kappa} \sqrt{Q} \]

\[ = P A(\kappa, \nu) \approx A \]
Where are we now?
3 Rank and Smoothness
Punchline

What do (numerical) rank and smoothness have to do with each other?
Recap: Multivariate Taylor

How does Taylor’s theorem get generalized to multiple dimensions?