Today

- Taylor: local, multipole, low rank
- Linear algebra
3 Rank and Smoothness
What do (numerical) rank and smoothness have to do with each other?
Recap: Multivariate Taylor

How does Taylor’s theorem get generalized to multiple dimensions?

\[ f(x) = G(x, y) \]

\[ G(x + h, y) = f(x + h) \approx \sum_{|\nu| \leq \kappa} \frac{D^\nu f(x)}{\nu!} h^\nu \]

\[ D^{(0, 0)} G(x, y) = \]

\[ D^{(1, 0)} G(x, y) = \partial_x G(x, y) \]

\[ \left| \frac{D^\nu G(x, y)}{\nu!} \right| \leq C \frac{1}{r^{1+\nu}} \left( |\nu|! \right)^{1/2} \]

\[ r = |x - y| \]
\[ \sum_{|\nu| \leq k} \frac{D^\nu G(x, y)}{\nu!} \quad \text{Taylor remainder} \]

\[ \leq \sum_{p=k+1}^{\infty} \alpha^p \in (\alpha^{k+1}) \]

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\[ \leq C \left( \frac{1}{\alpha} \right) \]

\[ \leq 2^r \]

\[ \leq \sum_{|\nu| \leq k} \frac{D^\nu G(x, y)}{\nu!} \quad \text{for all } \nu \]
\[ \leq C \left( \frac{\rho}{|v|} \right)^{|v|} \leq C' \left( \frac{\rho}{|v|} \right)^{|v|} \]

Computational tool:
- direct env. \( N_s N_t \)
- from Taylor \( k N_s \)
- eval Taylor \( k N_t \)

\[ \text{Error} \leq C \cdot \left( \frac{d(c, \text{farthest tgt})}{d(c, \text{closest src})} \right)^{k+1} \]
How many terms in the Taylor series?

\[ |v_1| + |v_2| \leq h \]

\[ \text{Rank: } \sim h^2 \]
Estimating the rank:

\[ e = \left( \frac{\text{dist}(c, \text{farthest hyd})}{\text{dist}(c, \text{closest src})} \right)^{\text{let} + 1} \]

\[ \text{rank} \propto k^2 \implies \sqrt{\text{rank}} = k \]

\[ e = \sqrt{\text{rank} + 1} \]

\[ k = \sqrt{\left( \frac{\log c}{\log s} - 1 \right)^2} \]
Taylor and Error

How can we estimate the error in a Taylor expansion?
## Connect Taylor and Low Rank

| Can Taylor help us establish low rank of an interaction? |
Compute a Taylor expansion of a 2D Laplace point potential.
Local expansions as a Computational Tool

Low rank makes evaluating interactions cheap(er). Do local expansions help with that goal?
Taylor on Potentials, Again

Stare at that Taylor formula again.