Today

Fast Multipole

Direct solving

PDE
Using Multipole-to-Local

Come up with an algorithm that computes the interaction in the figure.

\[ \text{Error: } \left( \frac{V_2}{3} \right)^{p+1} = \left( \frac{\partial f_s}{\partial c_t} \right)^{p+1} \]

(Figure credit: G. Martinsson, Boulder)
Using Multipole-to-Local

Assuming we retain information from the previous level, how can we obtain a valid local expansion on the target box?
Define ‘Interaction List’

For a box $b$, the interaction list $l_b$ consists of all boxes $b'$ so that
The Fast Multipole Method (‘FMM’)

**Upward pass**
1. Build tree
2. Compute interaction lists
3. Compute lowest-level multipoles from sources
4. Loop over levels \( \ell = L - 1, \ldots, 2 \):
   (a) Compute multipoles at level \( \ell \) by \( mp \rightarrow mp \)

**Downward pass**
1. Loop over levels \( \ell = 2, 3, \ldots, L - 1 \):
   (a) Loop over boxes \( b \) on level \( \ell \):
      i. Add contrib from \( I_b \) to local expansion by \( mp \rightarrow loc \)
      ii. Add contrib from parent to local exp by \( loc \rightarrow loc \)
2. Evaluate local expansion and direct contrib from 9 neighbors.

**Overall algorithm:** Now \( O(N) \) complexity.
Note: $L$ levels, numbered 0, …, $L - 1$. Loop indices above inclusive.
What about adaptivity?

Figure credit: Carrier et al. ('88)
What changes?
What about adaptivity?

Figure credit: Carrier et al. ('88)
Make a list of cases:
What about solving?

Likely computational goal: Solve a linear system $Ax = b$. How do our methods help with that?
A Matrix View of Low-Rank Interaction

Only parts of the matrix are low-rank! What does this look like from a matrix perspective?
Block-separable matrices

How do we represent the low-rank structure of a matrix like this?

\[ A = \begin{pmatrix} D_1 & A_{12} & A_{13} & A_{14} \\ A_{21} & D_2 & A_{23} & A_{24} \\ A_{31} & A_{32} & D_3 & A_{34} \\ A_{41} & A_{42} & A_{43} & D_4 \end{pmatrix} \]

where \( A_{ij} \) has low-rank structure?
Block-Separeable Matrices

A block-separable matrix looks like this:

\[
A = \begin{pmatrix}
D_1 & P_1 \tilde{A}_{12} \Pi_2 & P_1 \tilde{A}_{13} \Pi_3 & P_1 \tilde{A}_{14} \Pi_4 \\
\tilde{A}_{21} \Pi_1 & D_2 & P_2 \tilde{A}_{23} \Pi_3 & P_2 \tilde{A}_{24} \Pi_4 \\
\tilde{A}_{31} \Pi_1 & \tilde{A}_{32} \Pi_2 & D_3 & P_3 \tilde{A}_{34} \Pi_4 \\
\tilde{A}_{41} \Pi_1 & \tilde{A}_{42} \Pi_2 & \tilde{A}_{43} \Pi_3 & D_4
\end{pmatrix}
\]

Here:
- $\tilde{A}_{ij}$ smaller than $A_{ij}$
- $D_i$ has full rank (not necessarily diagonal)
- $P_i$ shared for entire row
- $\Pi_i$ shared for entire column

Q: Why is it called that?