Today:

- Low rank / smooth func.
- Taylor

↳ local expansion

multipole expansion

rank

"expansions" using LA

HW2

Smoothing Operators

If the operations you are considering are *smoothing*, you can expect to get a lot of mileage out of low-rank machinery.

What types of operations are smoothing?

Now: Consider some examples of smoothness, with justification.

How do we judge smoothness?

Recap: Multivariate Taylor

$$f(\vec{c}+\vec{h}) = \sum_{|\alpha| \leq p} \frac{D^\alpha f(\vec{c})}{\alpha!} h^\alpha$$

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$$

Taylor and Error (I)

How can we estimate the error in a Taylor expansion?

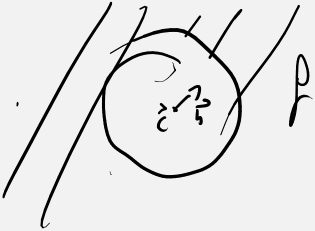
Taylor and Error (II)

Now suppose that we had an estimate that $\left|\frac{f^{(p)}(c)}{p!}h^p\right| \leqslant \alpha^p$.

$$\left|\sum_{p=k+1}^{\infty} \underbrace{\frac{f^{(p)}(c)}{p!} h^p}_{\alpha^p}\right| = \frac{1}{1-\alpha} \cdot \alpha^{k+1}$$

Connect Taylor and Low Rank

Can Taylor help us establish low rank of an interaction?



$$f(x) = f(c+h) \approx \sum_{|p|\le k} \frac{D^p(c)}{p!} \vec{h}^p$$

$$= \sum_{|p|\le k} (\text{coeff}) \cdot \varphi_p(h)$$

Taylor on Potentials (I)

Compute a Taylor expansion of a 2D Laplace point potential.

$$\psi(\vec{x}) = \sum_{i=1}^{n} G(\vec{x},\vec{y})\,\varphi(\vec{y}_i)$$

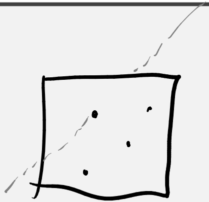
$$= \sum_{i=1}^{n} \log(\|\vec{x}-\vec{y}\|_2)\,\varphi(y_i)$$

$n=1 \quad \varphi(\vec{y}_1)=1$

Taylor exp. $G(\vec{x},\vec{y})$

Taylor on Potentials (Ia)

Why is it interesting to consider Taylor expansions of Laplace point potentials?



3D : $\frac{1}{r}$ 2D : $\log(\overset{\|x-y\|_2}{r})$

Taylor on Potentials (II)

$\vec{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ $r = \|\vec{d}\|_2$

$\partial_x \frac{1}{r} = \frac{d_1}{r^2}$

$d_2 \frac{1}{r} = \frac{d_2}{r^2}$

```
Maxima 5.42.1 http://maxima.sourceforge.net
(%i1) phi0: log(sqrt(y1**2 + y2**2));
                                   2     2
                             log(y2  + y1 )
(%o1)                        --------------
                                   2
(%i2) diff(phi0, y1);
                                  y1
(%o2)                          ---------
                                 2     2
                               y2  + y1
(%i3) diff(phi0, y1, 5);
                                    3              5
           120 y1          480 y1         384 y1
(%o3)   ------------ - ------------ + ------------
          2     2 3      2     2 4      2     2 5
        (y2  + y1 )    (y2  + y1 )    (y2  + y1 )
(%i4)
```

r^1 r^3 r^5 $\sim \frac{1}{r^5}$

r^6 r^8 r^{10}

Taylor on Potentials (III)

Which of these is the most dangerous (largest) term?
$\to$ Hard to say. They all contain the same number of powers of components of $\mathbf{y}$.

What's a bound on it? Let $R = \sqrt{y_1^2 + y_2^2}$.

$$\left| \frac{5040 y_1}{(y_2^2 + y_1^2)^4} \right| \leqslant C \left| \frac{y_1}{R^8} \right| \leqslant C \frac{1}{R^7}.$$

'Generalize' this bound:

$$|D^p \psi| \leqslant C_p \begin{cases} \log(R) & |p| = 0 \\ R^{-|p|} & |p| > 0 \end{cases}.$$

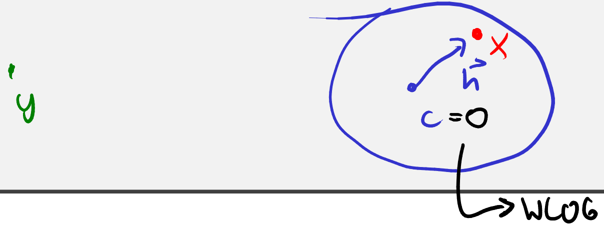
Appears true at least from the few p we tried. (Actually is true.)
C_p is a 'generic constant'–its value could change from one time it's written to the next.

Taylor on Potentials (IV)

$f(x,y) = f_1(x) f_2(y)$

What does this mean for the convergence of the Taylor series as a whole?

$$\left| \frac{D^p \psi(0) h^p}{p!} \right| \leq C_p \|y\|_2^{-p} \|h\|^p = C_p \left(\frac{\|h\|}{\|y\|} \right)^p$$



$$C_p\left(\frac{\|h\|}{\|y\|}\right)^p$$

$$\left| \sum G(x_i, y_i)\, \rho(y_i) - \left(\text{Taylor series of highest order } p\right) \right| \le C_p \left(\frac{d(c, \text{furthest target})}{d(c, \text{closest source})} \right)^{p+1}$$

Taylor on Potentials (V)

Lesson: As long as

$$\frac{\max_i |\mathbf{x}_i - \mathbf{c}|_2}{\min_j |\mathbf{y}_j - \mathbf{c}|_2} = \frac{r}{R} < 1,$$

the Taylor series converges.

Taylor on Potentials (VI)

A few remarks:

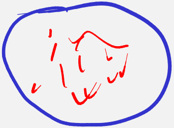
- We have just invented one specific example of what we will call a *local expansion* (of a potential ψ).
- The abstract idea of a *local expansion* is that:
  - it converges on the interior of a ball as long as the closest source is outside that ball,
  - The error in approximating the potential by a truncated (at order k) local expansion is

$$C_p \left(\frac{r}{R}\right)^{k+1} = \left(\frac{\mathrm{dist}(\mathbf{c}, \text{ furthest target})}{\mathrm{dist}(\mathbf{c}, \text{ closest source})}\right)^{k+1}$$

Taylor on Potentials: Low Rank?

Connect this to the numerical rank observations:

10^7



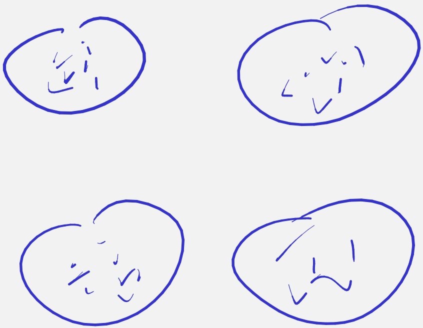
10^7

Taylor on Potentials: Low Rank

Low numerical rank is no longer a numerically observed oddity, it's mathematical fact.
Away from the sources, point potentials are smooth enough that their Taylor series ('local expansions') decay quickly. As a result, the potential is well-approximated by truncating those expansions, leading to low rank.

Local expansions as a Computational Tool

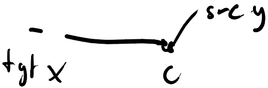
Low rank makes evaluating interactions cheap(er). Do local expansions help with that goal?



n galaxy clusters $\rightarrow$ still n^2

local exp to compute

Taylor on Potentials, Again



Stare at that Taylor formula again.

local $\rightarrow$ $$\psi(x-y) = \sum_{|p|\le k} \underbrace{\frac{D^p \psi(x-y)\big|_{x=c}}{p!}}_{\text{source}} \underbrace{(x-c)^p}_{\text{target}}$$

multipole $\rightarrow$ $$\psi(x-y) = \sum_{|p|\le k} \underbrace{\frac{D^p \psi(x-y)\big|_{y=c}}{p!}}_{\text{target}} \underbrace{(y-c)^p}_{\text{source}}$$

Multipole Expansions (I)

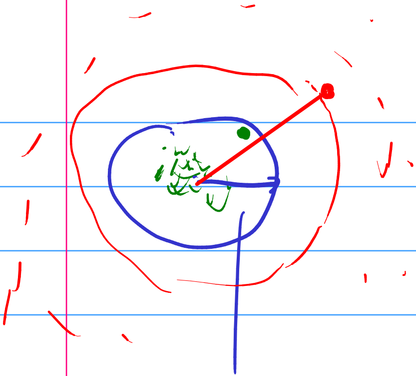
At first sight, it doesn't look like much happened, but mathematically/geometrically, this is a very different animal.

First Q: When does this expansion converge?

$$\left| \frac{D^p \psi(x-y)}{p!}\Big|_{y=c} (y-c)^p \right| \le C_p \left(\frac{\|y-c\|_2}{\|x-c\|_2} \right)^p$$

$$\left| \sum G(x_i, y_i)\, \rho(y_i) - (\text{Multipole } \cancel{\text{Taylor}} \text{ series of highest order } p) \right| \le C_{p+1} \left(\frac{d(c, \text{furthest src})}{d(c, \text{closest tgt})} \right)^{p+1}$$

Recall local case: $\left| \frac{\text{furthest tgt}}{\text{closest src}} \right|^p$



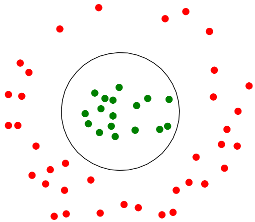
$$\frac{d(c, \text{furthest src})}{d(c, \text{closest tgt})}$$

Multipole Expansions (II)

The abstract idea of a *multipole expansion* is that:

- it converges on the exterior of a ball as long as the furthest source is closer to the center than the closest target,
- The error in approximating the potential by a truncated (at order k) local expansion is

$$\left(\frac{\operatorname{dist}(\mathbf{c}, \text{furthest source})}{\operatorname{dist}(\mathbf{c}, \text{closest target})}\right)^{k+1}.$$



The multipole expansion converges everywhere outside the circle!
(Possibly: slowly, if the targets are too close–but it does!)