Today

- Multipoles
- Rank estimates
- Multipoles and locals using LA
- Near and far

Taylor on Potentials, Again

Stare at that Taylor formula again.

$$\Psi(x) \approx \sum_{|p| \le k} \frac{D^p \Psi(x-y)|_{x=c}}{p!} (x-c)^p$$

$$\Psi(x) \approx \sum_{|p| \le k} \frac{D^p \Psi(x-y)|_{y=c}}{p!} (y-c)^p$$

Multipole Expansions (I)

At first sight, it doesn't look like much happened, but mathematically/geometrically, this is a very different animal.

First Q: When does this expansion converge?

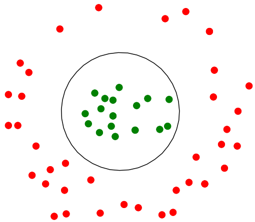
$$|\text{Err in Multipole exp.}| \leq \left(\frac{d(c, \text{Furthest src})}{d(c, \text{closest target})}\right)^{p+1}$$

Multipole Expansions (II)

The abstract idea of a *multipole expansion* is that:

- it converges on the exterior of a ball as long as the furthest source is closer to the center than the closest target,
- The error in approximating the potential by a truncated (at order k) local expansion is

$$\left(\frac{\text{dist}(\mathbf{c}, \text{ furthest source})}{\text{dist}(\mathbf{c}, \text{ closest target})}\right)^{k+1}.$$



The multipole expansion converges everywhere outside the circle!
(Possibly: slowly, if the targets are too close–but it does!)

Dipole?

$\overset{\bullet}{+} \quad \overset{\bullet}{-}$

$$\lim_{\delta \to 0} \frac{G(x+\delta) - G(x-\delta)}{2\delta} = \frac{d}{dx} G$$

Multipole Expansions (III)

If our particle distribution is like in the figure, then a multipole expansion is a computationally useful thing. If we set

- S = #sources,
- T = #targets,
- K = #terms in expansion,

then the cost *without* the expansion is $O(ST)$, whereas the cost *with* the expansion is $O(SK + KT)$.
If $K \ll S, T$, then that's going from $O(N^2)$ to $O(N)$.

The rank (#terms) of the multipole expansion is the same as above for the local expansion.

Demo: Multipole/local expansions

Taylor on Potentials: Low Rank?

Connect this to the numerical rank observations:

How many items in $\sum_{|p|\le k} \cdots$

1D: $k+1$

2D: $\frac{(k+1)(k+2)}{2}$

3D: $\frac{(k+1)(k+2)(k+3)}{6}$

$= O(k^d)$

$p=(p_1,p_2)$

$p_1+p_2\le k$

$$|\text{Error in local}| \simeq \left(\frac{d(c, \text{furthest tgt})}{d(c, \text{closest src})} \right)^{k+1}$$

On Rank Estimates

So how many terms do we need for a given precision ε?

$$\varepsilon \approx \left(\frac{d(c, \text{furthest tgt})}{d(c, \text{closest src})} \right)^{k+1} = \rho^{k+1}$$

items → K

order → k

2D $\quad K \approx k^2 \quad k = \sqrt{K}$

$$\varepsilon \approx \rho^{\sqrt{K}+1}$$

$$\frac{\log \varepsilon}{\log \rho} \approx (\sqrt{K}+1)$$

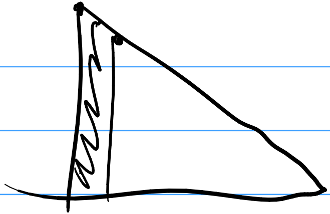
$$K \approx \left(\frac{\log \varepsilon}{\log \rho} - 1 \right)^2$$

Demo: Checking rank estimates

$f \quad \partial_x f \quad \partial_x^2 f$

$\partial_y f$

$\partial_y^2 f$



$\log \longrightarrow f \qquad \Delta f = 0$

$$\partial_x^2 f + \partial_y^2 f = 0$$

$$\partial_x^2 = -\partial_y^2$$

Estimated vs Actual Rank

Our rank estimate was off by a power of $\log \varepsilon$. What gives?

PDE

Taylor and PDEs

Look at $\partial_x^2 G$ and $\partial_y^2 G$ in the multipole demo again. Notice anything?

$\partial_x^2 = -\partial_y^2$

Being Clever about Expansions

How could one be clever about expansions? (i.e. give examples)

$\to f: \mathbb{C} \to \mathbb{C}$ differentiable

$\Delta \operatorname{Re} f = 0$

$f'(x) \cdot h = f(x+h) - f(x)$

$\rightsquigarrow$ Cauchy-Riemann PDEs

$\hookrightarrow$ complex Taylor expansion

Expansions for Helmholtz

$\Delta u = 0$ Laplace

$\Delta u + k^2 u = 0$ Hh.

How do expansions for other PDEs arise?

Do sep. of variables in polar coordinates

↪ ODE in r

DLMF 10.23.6 shows 'Graf's addition theorem':

$$H_0^{(1)}(\kappa \|x - y\|_2) =$$

$$\sum_{\ell=-\infty}^{\infty} \underbrace{H_\ell^{(1)}(\kappa \|y - c\|_2)\, e^{i\ell\theta'}}_{\text{singular}} \underbrace{J_\ell(\kappa \|x - c\|_2)\, e^{-i\ell\theta}}_{\text{nonsingular}}$$

where $\theta = \angle(x - c)$ and $\theta' = \angle(x' - c)$.

Can apply same family of tricks as with Taylor to derive multipole/local expansions.

Making Multipole/Local Expansions using Linear Algebra

Actual expansions cheaper than LA approaches. Can this be fixed?

Compare costs for this situation:

S sources T targets

Form interaction matrix: ST

The Proxy Trick

Idea: *Skeletonization using Proxies*

Demo: Skeletonization using Proxies

Q: What error do we expect from the proxy-based multipole/local 'expansions'?