Today

- Spurious res. in ext. problems
- Calderon identities
- Numerics

Announcements

- Presentation dates ✓
- HW4: due next week

	First slot	Second slot
Dec 4	Guanhua	Yashraj
Dec 6	Hongliang	Vedant
Dec 11	Jonathan	Yucha

Resonances

$-\triangle$ on a bounded (interior) domain with homogeneous Dirichlet/Neumann BCs has countably many real, positive eigenvalues.

What does that have to with Helmholtz?

Why could it cause grief?

Helmholtz: Boundary Value Problems

Find $u \in C(\bar{D})$ with $\triangle u + k^2 = 0$ such that

	Dirichlet	**Neumann**
Int.	$\lim_{x \to \partial D -} u(x) = g$ unique ($-$resonances)	$\lim_{x \to \partial D -} \hat{n} \cdot \nabla u(x) = g$ unique ($-$resonances)
Ext.	$\lim_{x \to \partial D +} u(x) = g$ Sommerfeld unique	$\lim_{x \to \partial D +} \hat{n} \cdot \nabla u(x) = g$ Sommerfeld unique

with $g \in C(\partial D)$.

Find layer potential representations for each.

Patching up resonances

Issue: Ext. IE inherits non-uniqueness from 'adjoint' int. BVP

Fix: Tweak representation [Brakhage/Werner '65, ...]
(also called the *CFIE* or *combined field integral equation*)

$$u = D\phi - i\alpha S\phi$$

(α: tuning knob $\to 1$ is fine, $\sim k$ better for large k)

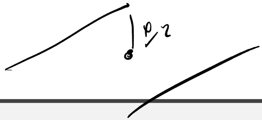
Patching up resonances: CFIE (1/3)

$u = D\varphi - iS\varphi$

Suppose:

$\underline{\varphi/2} + D\varphi - iS\varphi = 0$. To show: $\varphi = 0$.

Patching up resonances: CFIE (2/3)



$$u|_{\mathbb{R}^n \setminus \Omega} = 0 \quad , \quad \lim_{+} \partial_n u = 0,$$

$$0 - (\partial_n u)^- = [\partial_n u] = [\partial_n (D\varphi - iS\varphi)] = i\varphi$$

$$0 - u^- = [u] = [D\varphi - iS\varphi] = \varphi$$

$$-i\,(\partial_n u)^- = u^-$$

Patching up resonances: CFIE (3/3)

$$-i\,(\partial_n u)^- = u^-$$

$v = \bar{u}$

$$\int_\Omega u\Delta v + \nabla u \cdot \nabla v = \int_\partial u\,\partial_n v$$

$$\underbrace{\int_\Omega -k|u|^2 + |\nabla u|^2}_{\in\mathbb{R}} = \int_\Omega u\Delta\bar{u} + |\nabla u|^2 = \int_\partial u^-(\partial_n \bar{u})^-$$

$$= \underbrace{-i\underbrace{\int_\partial |u^-|^2}_{\in\mathbb{R}}}_{\in i\mathbb{R}}$$

Want: $u^- = 0$. ✓

$u^+ = 0 \quad \mathcal{O}[u] = \varphi$ ✓

- Burkhardt Werner
- Panich
- ?

$$-k u\bar{u} = -k|u|^2 \quad \nabla u \cdot \nabla \bar{u} = |\nabla u|^2$$

Helmholtz Uniqueness

Uniqueness for remaining IEs similar:

✓

A word about D'

$$(\varphi, D'\psi) = ((D')^*\varphi, \psi)$$

Show that D' is self-adjoint. [Kress LIE 3rd ed. Sec 7.6]

Green's second

$$\underbrace{\int u\Delta v - v\Delta u}_{0} = \int_\partial (u\partial_n v) - (v\partial_n u)$$

$$u = D\varphi \qquad v = D\psi$$

$$\int u\partial_n v = \int v\partial_n u$$

$$\longrightarrow (u, \partial_n v) = (v, \partial_n u)$$

$$(D'\varphi, \psi)$$

$$= (\partial_n u, [v])$$

$$= (\partial_n u^+, v^+) - (\partial_n u^-, v^-) = (u^+, (\partial_n v)^+) - (u^-, (\partial_n v)^-)$$

$$= (\varphi, D'\psi).$$

Towards Calderón

Show that $(S\varphi, D'\psi) = ((S' + I/2)\varphi, (D - I/2)\psi)$.

$$(S\varphi, D'\psi) = (w, \partial_n v) = (\partial_n w, v)$$

$$\overset{\uparrow}{w} \quad v = D\psi \qquad = ((S' + \tfrac{I}{2})\varphi, (D - \tfrac{I}{2})\psi)$$

$(\varphi, SD'\psi)$?

$$(\varphi, SD'\psi) = (S\varphi, D'\psi)$$

$$= ((S' + \tfrac{I}{2})\varphi, (D - \tfrac{I}{2})\psi)$$

$$= (\varphi, (D + \tfrac{I}{2})(D - \tfrac{I}{2})\psi)$$

$$= (\varphi, (D^2 - \tfrac{I}{4})\psi)$$

Calderón Identities: Summary

- $SD' = D^2 - I/4$
- $D'S = S'^2 - I/4$

Also valid for Laplace (jump relation same after all!)

Why do we care?

Original CFIE:

$D\varphi - iS\varphi$

$DS\varphi - iS\varphi$

Ext. Neumann IE:

$\frac{\varphi}{2} + D'\varphi - iS'\varphi = \partial_n \cdots$

$\frac{\varphi}{2} + D'S\varphi - iS'\varphi$

Outline

Numerics: What do we need?

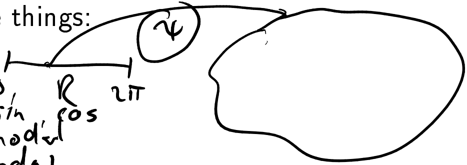
- Discretize curves and surfaces
  - Interpolation
  - Grid management
  - Adaptivity
- Discretize densities
- Discretize integral equations
  - Nyström, Collocation, Galerkin
- Compute integrals on them
  - "Smooth" quadrature
  - Singular quadrature
- Solve linear systems

Constructing Discrete Function Spaces

Floating point numbers (*Degrees of Freedom* or *DoFs*) $\leftrightarrow$ Functions

Discretization relies on three things:

- Base/reference domain
- Basis of functions
- Meaning of DoFs $<$ modal / nodal



Related finite element concept: *Ciarlet triple*

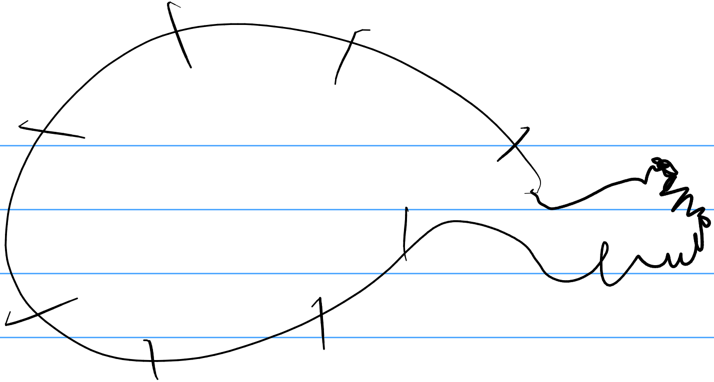
$\psi : R \to \mathbb{R}^2$

Discretization options for a curve?

worse → sub

ψ same discr. as density → isoparametric discr.

better → super



not good for Fourier

What do the DoFs mean?

Common DoF choices:

- Point values of function
- Point values of (directional?) derivatives
- Basis coefficients
- Moments

Often: useful to have both “modes”, “nodes”, jump back and forth