Fast Algorithms and Integral Equation Methods

CS598 APK

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Today:

- Sales pitch

Outline

What's the point of this class?

- Starting point: Large-scale scientific computing
- Many popular numerical algorithms: $O(n^\alpha)$ for $\alpha > 1$
  (Think Matvec, Matmat, Gaussian Elimination, LU, ...)
- Build a set of tools that lets you cheat: Keep α small
  (Generally: probably not–Special purpose: possible!)
- Final goal: Extend this technology to yield PDE solvers
- But: Technology applies in many other situations
  - Many-body simulation
  - Stochastic Modeling
  - Image Processing
  - 'Data Science' (e.g. Graph Problems)
- This is class is about an even mix of math and computation

Survey

- Home dept
- Degree pursued
- Longest program ever written
  - in Python?
- Research area
- Interest in PDE solvers

Class web page

https://bit.ly/fastalg-f19

contains:

- Class outline
- Assignments
- Piazza
- Grading
- Video

– Interactive demos!

Why study this at all?

- Finite difference/element methods are inherently
  - ill-conditioned
  - tricky to get high accuracy with
- Build up a toolset that does *not* have these flaws
- Plus: An interesting/different analytical and computational point of view
  - If you're not going to use it to solve PDEs, it (or the ideas behind it) will still help you gain insight.

FD/FEM: Issues

Idea of these methods:

1. Take differential equations
2. Discretize derivatives
3. Make linear system
4. Solve

So what's wrong with doing that?

Discretizing Derivatives: Issues?

$$(e^{i\alpha x})' = i\alpha e^{i\alpha x} \rightarrow$$

$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|$$

→ derivatives have a nullspace

→ derivatives give rise to unbounded operators

e^{inx}

Discretizing Derivatives: Issues?

Result: The better we discretize (the more points we use), the worse the condition number gets.

Demo: Conditioning of Derivative Matrices

To be fair: Multigrid works around that (by judiciously using fewer points!)

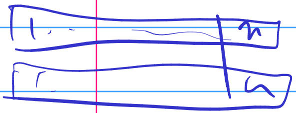
But there's another issue that's not fixable.

Q: Are these problems real?

So this class is about starting fresh with methods that (rigorously!) don't have these flaws!

$$i \quad \begin{pmatrix} \varphi_j(x_i) & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix}$$

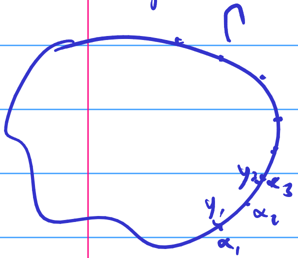
$$\begin{pmatrix} \varphi'_j(x_i) & & \\ & & \\ & & \end{pmatrix}$$



Sketch: How do IEs work?

$\Delta u = 0$, $u = g$ on Γ

fundamental solution $\Delta G = \alpha_j \delta(x - y_j)$



$\Delta u = \rho \to G'$

$E = -\nabla u \qquad \operatorname{div} E = \rho$

$$u_n(x) = \sum_{i=1}^{n} G(x - y_i)\,\alpha_i$$

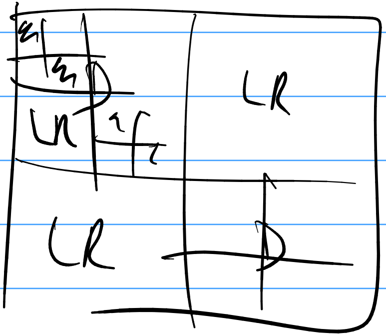
$$u(x) = \int_\Gamma G(x - y)\,\alpha(y)\,dy$$

layer potentials

— and what's wrong with them?

Problems:

- Singular quadrature → Yuck!
  - ↳ Near-
- Dense matrices?
- Ill-conditioned ←
  - ↳ $\frac{1}{2}I + D$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Bonus Advertising Goodie

Both multigrid and fast/IE schemes ultimately are $O(N)$ in the number of degrees of freedom N.

Outline