

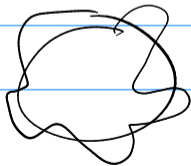
Today

- Butterfly
- IEs for PDE solving
- Theory

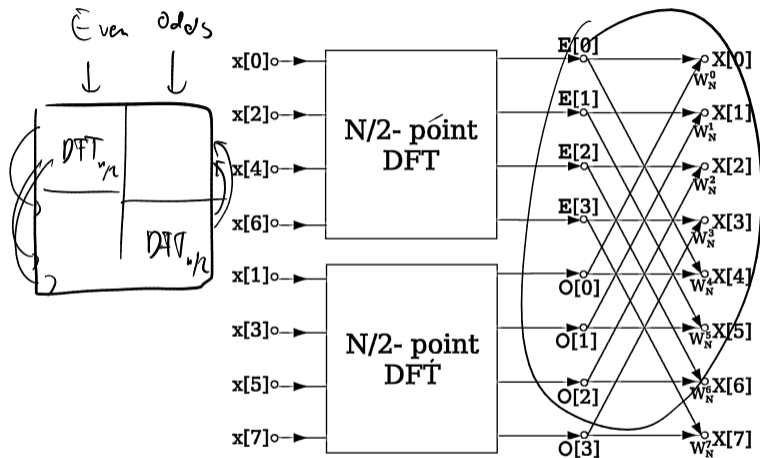
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Announcements

- HW3
- Projects
- Office hour moved to 4:30



# FFT: Data Flow



Perhaps a little bit like a butterfly?

# Fourier Transforms: A Different View

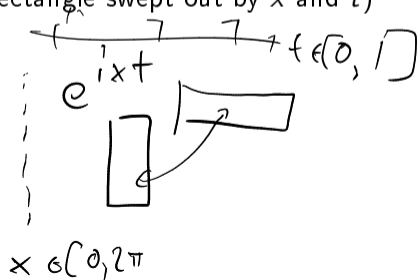
Claim:

*The [numerical] rank of the normalized Fourier transform with kernel  $e^{ixt}$  is bounded by a constant times  $\gamma$ , at any fixed precision  $\epsilon$ .*

(i.e. rank is bounded by the area of the rectangle swept out by  $x$  and  $t$ )

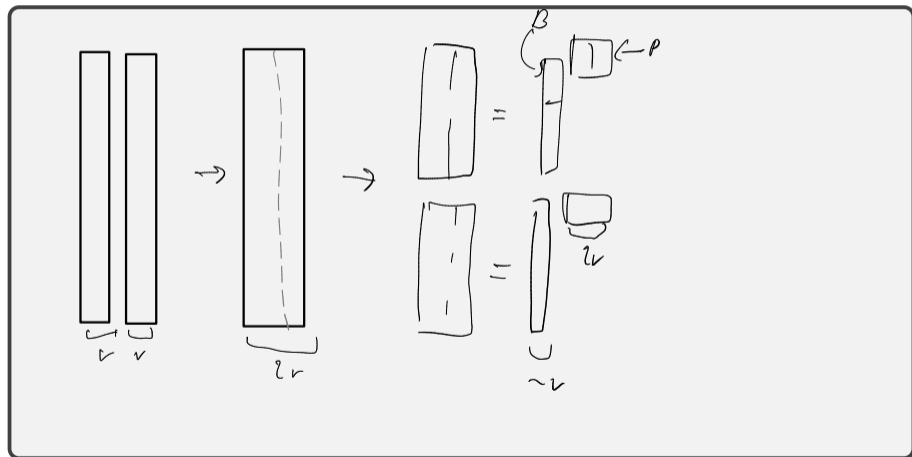
[[O'Neil et al. '10](#)]

**Demo: Butterfly Factorization** (Part I)



## Recompression: Making use of Area-Bounded Rank

How do rectangular submatrices get expressed so as to reveal their constant rank?



## Observations

### Demo: Butterfly Factorization (Part II)

For which types of matrices is the Butterfly factorization guaranteed accurate?

all of them

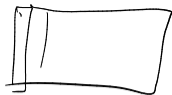
For which types of  $n \times n$  matrices does the butterfly lead to a reduction in cost?

rect prop

Explore the limit cases of the characterization.

$\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} ?$

## Observations: Cost



What is the cost (in the reduced-cost case) of the matvec?

$$\text{Level 0: } P_{0,k} \quad \underbrace{r \times \underbrace{n/2^k}}_{}$$

$$\text{Level } k: P_{k,i,j} \quad r \times 2r$$

$$\text{Postroc. } B_{i,j} \quad n/2^k \times r$$

Comments?

# Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

**Outlook: Building a Fast PDE Solver**

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

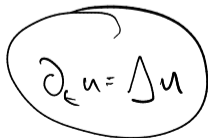
# PDEs: Simple Ones First, More Complicated Ones Later

## Laplace

$$\underline{\Delta} u = 0$$

$$\Delta = \sum_{k=1}^n \partial_{x_k}^2$$

- ▶ Steady-state  $\partial_t u = 0$  of wave propagation, heat conduction
- ▶ Electric potential  $u$  for applied voltage
- ▶ Minimal surfaces/“soap films”
- ▶  $\nabla u$  as velocity of incompressible flow


$$\partial_\epsilon u = \Delta u$$

## Helmholtz

$$\Delta u + k^2 u = 0$$

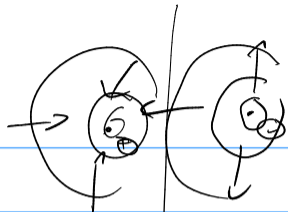
- ▶ Assume time-harmonic behavior  $\tilde{u} = e^{\pm i\omega t} u(x)$  in time-domain wave equation:

$$\partial_t^2 \tilde{u} = \Delta \tilde{u}$$

- ▶ Sign in  $\tilde{u}$  determines direction of wave: Incoming/outgoing if free-space problem
- ▶ *Applications:* Propagation of sound, electromagnetic waves



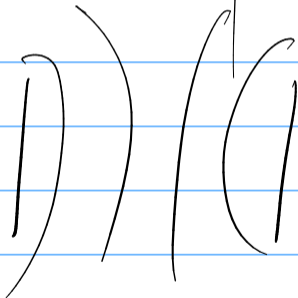
$$\nabla \cdot \mathbf{E} = \rho$$



$$\mathbf{E} = \nabla u$$

$$\nabla \cdot \nabla u = \rho$$

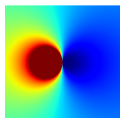
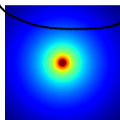
$$\uparrow$$
$$0$$



## Fundamental Solutions

Laplace

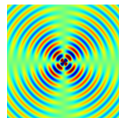
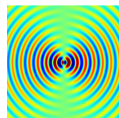
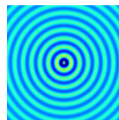
$$-\Delta u = \delta$$



$$\Delta u = f$$

Helmholtz

$$\Delta u + k^2 u = \delta$$



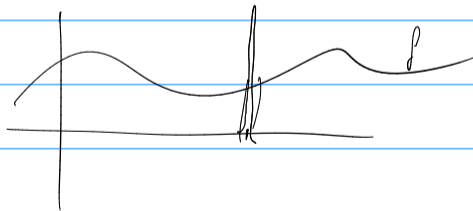
aka. *Free space Green's Functions*

How do you assign a precise meaning to the statement with the  $\delta$ -function?

$$\Delta u = \delta \rightarrow \int \Delta u \varphi = \int \delta \varphi$$

$$f(x) \rightarrow$$

$$\delta(x) \rightarrow \int \delta(x) \varphi(x) dx$$



## Green's Functions

Why care about Green's functions?

.

What is a non-free-space Green's function? I.e. one for a specific domain?

## Green's Functions (II)

Why not just use domain Green's functions?

A large, empty rounded rectangular box with a thin black border, intended for a user to provide an answer to the question above.

What if we don't know a Green's function for our PDE... at all?

A large, empty rounded rectangular box with a thin black border, intended for a user to provide an answer to the question above.