$(1-A)\varphi = f$

Today

- Riesz
- Fredholm
- Spectral

Announcements

- HW4 (Mat of 1op, ~~Barnes Hut~~ BSS solve, QBX)
- Project prop.

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n]$ such that

$$|K(x,y)| \leq C|x-y|^{\alpha-n} \qquad (x \neq y)$$

Theorem (Weakly singular kernel $\Rightarrow$ compact [Kress LIE 2nd ed. Thm. 2.22])

K weakly singular. Then

$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

"a volume domain"

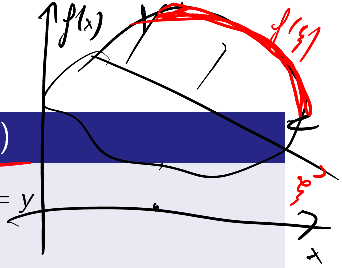
is compact on $C(G)$.

Weakly singular: Proof Outline

Outline the proof of 'Weakly singular kernel $\Rightarrow$ compact'.

Weakly singular (on surfaces)

$\Omega \subset \mathbb{R}^n$ bounded, open, $\partial\Omega \in C^1$



Definition (Weakly singular kernel (on a surface))

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n-1]$ such that

$$|K(x,y)| \leq C|x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

Theorem (Weakly singular kernel $\Rightarrow$ compact [Kress LIE 2nd ed. Thm. 2.23])

K weakly singular on $\partial\Omega$. Then $(A\phi)(x) := \int_{\partial\Omega} K(x,y)\phi(y)dy$ is compact on $C(\partial\Omega)$.

Riesz Theory (I)

Still trying to solve

$$L\phi := (I - A)\phi = \phi - A\phi = f$$

with A compact.

Theorem (First Riesz Theorem [Kress, Thm. 3.1])

$N(L)$ is finite-dimensional.

Questions:

- What is $N(L)$ again?
- Why is this good news?

$\varphi \in N(L) \Leftrightarrow L\varphi = 0$

$A\varphi = \varphi$

$A|_{N(L)} = Id$

$\Rightarrow N(L)$ finite-dim

Riesz First Theorem: Proof Outline

Show it.

Riesz Theory (II)

Theorem (Riesz theory [Kress, Thm. 3.4])

A compact. Then:

- $(I - A)$ *injective* $\Leftrightarrow$ $(I - A)$ *surjective*
    - *It's either bijective or neither s nor i.*
- *If $(I - A)$ is bijective, $(I - A)^{-1}$ is bounded.*

Rephrase for solvability:

$(I-A)\varphi = \varphi - A\varphi = f$ is solvable if $N(L) = \{0\} \Leftrightarrow \varphi - A\varphi = 0$ only has the trivial solution.

Key shortcoming?

Goes away if $N(L) \neq \{0\}$.

Suppose C^{-1} unbounded, then $\exists f_n$, $\|f_n\|=1$, $\|C^{-1}f_n\| \geq n$

$$g_n = \frac{f_n}{\underbrace{\|C^{-1}f_n\|}_{\to 0}} \qquad \varphi_n = \frac{C^{-1}f_n}{\|C^{-1}f_n\|} \qquad \|\varphi_n\| = 1$$

$$\varphi_n - A\varphi_n = g_n \qquad \varphi_{n(k)} : A\varphi_{n(k)} \to \varphi$$

$$\varphi_n = C^{-1}g_n \Leftrightarrow C\varphi_n = g_n$$

$$\varphi_{n(k)} - A\varphi_{n(k)} = g_{n(k)} \to 0$$

$$\varphi_{n(k)} \to \varphi, \quad A\varphi_{n(k)} \to \psi$$

$$\varphi_n \to \varphi$$

$$\varphi \in N(L) \iff (\varphi - A\varphi) = 0$$

$$\Downarrow$$

$\varphi = 0$ (assumed trivial nullspace, because L onto)

Hilbert spaces

Hilbert space: Banach space with a norm coming from an *inner product*:

$$(\alpha x + \beta y, z) = ?$$

$$(x, \alpha y + \beta z) = ?$$

$$\sqrt{(x, x)} = \|x\|$$

$$(y, x) = ?$$

Is $C^0(G)$ (with $\|\cdot\|_\infty$) a Hilbert space?

no

Name a Hilbert space of functions.

$L^2(\Omega) = \{ \int_\Omega f^2 \, dx < \infty \}$

Continuous and Square-Integrable

Is $C^0(G)$ “equivalent” to $L^2(G)$? $|(f,f)| \leq \|f\|_\infty$

Why do compactness results transfer over nonetheless? Hint: What is

$$|(x,y)| \leq ? \|x\| \|y\|$$

Adjoint Operators

Definition (Adjoint oeprator)

A^* called adjoint to A if

$$(Ax, y) = (x, A^* y)$$

for all x, y.

Facts:

- A^* unique
- A^* exists
- A^* linear
- A bounded $\Rightarrow$ A^* bounded
- A compact $\Rightarrow$ A^* compact

Adjoint Operator: Observations?

What is the adjoint operator in finite dimensions? (in matrix representation)

transpose $\quad y_j = \sum A_{ij} x_i$

What do you expect to happen with integral operators?

swap targets and sources $\quad \varphi(x) = \int k(x,y)\, \sigma(y)\, dy$

Adjoint of the single-layer?

$S\sigma(x) = \int_\Gamma \frac{1}{|x-y|} \sigma(y)\, dy \qquad S^* = S$

Adjoint of the double-layer?

$D\sigma(x) = \int_\Gamma \underbrace{\partial_{n_y} \frac{1}{|x-y|}} \sigma(y)\, dy \quad \Big| \quad S'\sigma(x) = \partial_{n_x} \int_\Gamma \frac{1}{|x-y|} \sigma(y)\, dy$

Fredholm Alternative

Theorem (Fredholm Alternative [Kress LIE 2nd ed. Thm. 4.14])

$A : X \to X$ *compact. Then either:*

- $I - A$ *and* $I - A^*$ *are bijective* $\leftarrow$

  or:

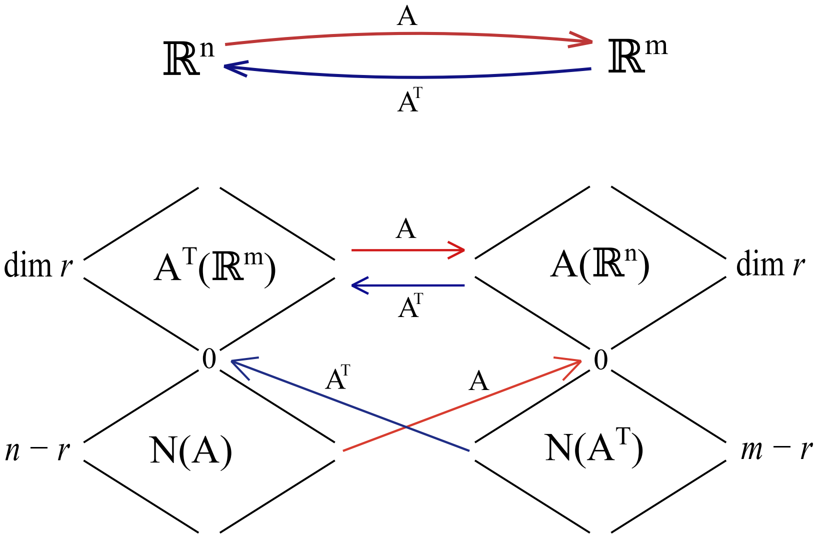
- $\dim N(I - A) = \dim N(I - A^*)$
- $(I - A)(X) = N(I - A^*)^\perp$ $\leftarrow$
- $(I - A^*)(X) = N(I - A)^\perp$

$(I - A)^* = I - A^*$

Seen these statements before?

Fundamental thm. of linear algebra

Fundamental Theorem of Linear Algebra



Fredholm Alternative in IE terms

Translate to language of integral equation solvability:

If $\varphi - A\varphi = 0 \Rightarrow \varphi = 0 \Rightarrow \varphi - A\varphi = f$ <u>has</u> a unique soln.

$\varphi(x) - \int k(x,y)\,\varphi(y) = f(x)$ is solvable iff

$f \perp \psi$ for all solutions ψ of

$\psi(x) - \int k(y,x)\,\psi(y)\,dy = 0.$

$(I - A)(X) = N(I - A^*)^\perp$

Fredholm Alternative: Further Thoughts

What about symmetric kernels $(K(x, y) = K(y, x))$?

$A = A^*$

Where to get uniqueness?

$\varphi - \mathcal{D}\varphi = 0 \Rightarrow \varphi$

Spectral Theory: Terminology

$A : X \to X$ bounded, λ is a ... value:

Definition (Eigenvalue)

There exists an element $\phi \in X$, $\phi \neq 0$ with $A\phi = \lambda\phi$.

Definition (Regular value)

The "resolvent" $(\lambda I - A)^{-1}$ exists and is bounded.

Can a value be regular and "eigen" at the same time?

$A\varphi - \lambda\varphi = 0$ no.

What's special about ∞-dim here?

Not all non-regular values are eigen.

Resolvent Set and Spectrum

Definition (Resolvent set)

$\rho(A) := \{\lambda \text{ is regular}\}$

Definition (Spectrum)

$\sigma(A) := \mathbb{C} \setminus \rho(A)$

Spectral Theory of Compact Operators

$A\varphi = f$

Theorem

$A : X \to X$ compact linear operator, X ∞-dim.

Then:

- $0 \in \sigma(A)$ *(show!)*
- $\sigma(A) \setminus \{0\}$ *consists only of eigenvalues*
- $\sigma(A) \setminus \{0\}$ *is at most countable*
- $\sigma(A)$ *has no accumulation point except for 0*

Spectral Theory of Compact Operators: Proofs

Show first part.

Show second part.