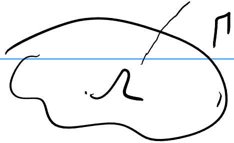
Today

- $\Delta u = 0$ ← Green's formula on unbdd.
- Layer pot: | jump relations
- BVP: uniqueness
  IEs
  existence

$$D1(x) = -1_\Omega(x)$$

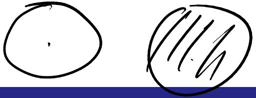
Announcements

- Projects
- HW4



$$D\varphi = \int_\Gamma \partial_{n_y} G(x,y)\varphi(y)\, ds_y$$

Mean Value Theorem



Theorem (Mean Value Theorem [Kress LIE 2nd ed. Thm 6.7])

If $\Delta u = 0$, $u(x) = \overline{\int}_{B(x,r)} u(y)dy = \overline{\int}_{\partial B(x,r)} u(y)dy$

Define $\overline{\int}$?

$-\frac{1}{r}$

$$\int G(x,y)\,\partial_n u(y)\,dS_y$$

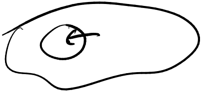
$v = G$

Trace back to Green's Formula (say, in 2D):

$$\int \partial_n u\, v - \int u\, \partial_n v$$

$$S(\partial_n u) - D(u) = \frac{\log r}{2\pi} \underbrace{\int_{\partial B(x,r)} \partial_n u \, dS_y}_{0} - \underbrace{\frac{\partial_r \log(r)}{2\pi}}_{+\frac{1}{2\pi r}} \int_{\partial B(x,r)} u(y)\, dS_y$$

Maximum Principle



Theorem (Maximum Principle [Kress LIE 2nd ed. 6.9])

If $\triangle u = 0$ on compact set $\bar{\Omega}$:
u attains its maximum on the boundary.

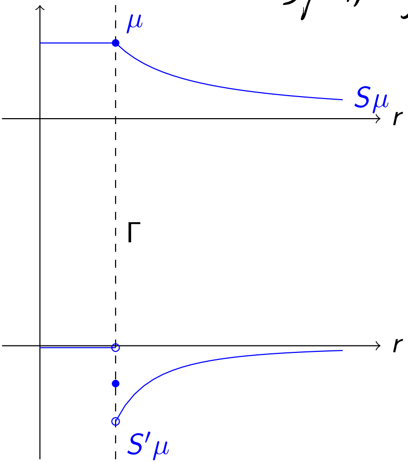
Suppose it were to attain its maximum somewhere inside an open set...

that contradicts the mean value thm.

What do our *constructed* harmonic functions (layer potentials) do there?

Jump relations:

$$S\mu(x) = \int_\Gamma \frac{\log|x-y|}{2\pi}\,\mu(y)\,dy$$



Jump Relations: Mathematical Statement

$D1(x) = 1_\Omega(x)$

Let $[X] = X_+ - X_-$. (Normal points towards "+"="exterior".)

Theorem (Jump Relations [Kress LIE 2nd ed. Thm. 6.14, 6.17,6.18])

$$[S\sigma] = 0$$

$$\lim_{x\to x_0\pm}(S'\sigma) = \left(S' \mp \frac{1}{2}I\right)(\sigma)(x_0) \quad\Rightarrow\quad [S'\sigma] = -\sigma$$

$$\lim_{x\to x_0\pm}(D\sigma) = \left(D \pm \frac{1}{2}I\right)(\sigma)(x_0) \quad\Rightarrow\quad [D\sigma] = \sigma$$

"hypersingular" $\rightarrow$ $[D'\sigma] = 0$

Truth in advertising: Assumptions on Γ?

C^2

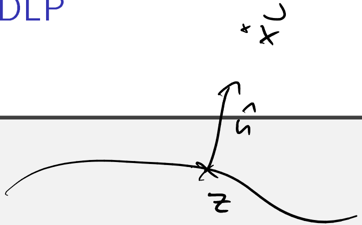
Jump Relations: Proof Sketch for SLP

Sketch the proof for the single layer.

uniformly conv. seq of cont. functions

Jump Relations: Proof Sketch for DLP

Sketch proof for the double layer.



$$D\sigma(x) = \sigma(z)(D1)(x) + D[\underbrace{\sigma - \sigma(z)}_{\to 0}]$$

$$x \to z+$$

Green's Formula at Infinity: Motivation

$\Omega \subseteq \mathbb{R}^n$ bounded, C^1, connected boundary, $\triangle u = 0$ in $\mathbb{R}^n \setminus \Omega$, u bounded

$$\boxed{(S_{\partial\Omega}(\hat{n}\cdot\nabla u) - D_{\partial\Omega}u)(x)} + \underbrace{(S_{\partial B_r}(\hat{n}\cdot\nabla u) - D_{\partial B_r}u)(x)}_{u_\infty} = u(x)$$

for x between $\partial\Omega$ and $B(x,r)$.

Behavior of individual terms? (2D)

- $S_{\partial B_r(x)}(\partial_n u) = \int_{\partial B_r(x)} \log|x-y|\,|\partial_n u|\,dS_y = \log r \int_{\partial B_r(x)} |\partial_n u|\,dS_y$
- $|D_{\partial B_r}(u)| = \left|\int_{\partial B_r(x)} \frac{x-y}{|x-y|^2}\, u\,dS_y\right| \le \frac{1}{r}\int_{\partial B_r(x)} |u|$

Green's Formula at Infinity: Statement

u bounded, harmonic in $\mathbb{R}^n \setminus \Omega$

Theorem (Green's Formula in the exterior [Kress LIE 3rd ed. Thm 6.11])

$$(S_{\partial\Omega}(\hat{n} \cdot \nabla u) - D_{\partial\Omega} u)(x) + \mathrm{PV} u_\infty = u(x)$$

for some constant u_∞. Only for $n = 2$,

$$u_\infty = \frac{1}{2\pi r} \int_{|y|=r} u(y) ds_y.$$

Realize the power of this statement:

u, $\partial_n u$ on $\partial\Omega$ are potentially sufficient to represent u.

Green's Formula at Infinity: Impact

Can we use this to bound u as $x \to \infty$?

Consider the behavior of the fundamental solution as $r \to \infty$.

$$u(x) = u_\infty + O\left(\frac{1}{|x|}\right)$$

How about u's derivatives?

$$\nabla u(x) = O\left(\frac{1}{|x|^{n-1}}\right)$$

Outline

Boundary Value Problems: Overview

$\Delta u = 0 \quad u = g$

	Dirichlet	Neumann
Int.	$\lim_{x\to\partial\Omega-} u(x) = g$ unique	$\lim_{x\to\partial\Omega-} \hat{n}\cdot\nabla u(x) = g$ may differ by constant
Ext.	$\lim_{x\to\partial\Omega+} u(x) = g$ $u(x) = \begin{cases} O(1) & 2D \\ o(1) & 3D \end{cases}$ as $\lvert x\rvert \to \infty$ unique	$\lim_{x\to\partial\Omega+} \hat{n}\cdot\nabla u(x) = g$ $u(x) = o(1)$ as $\lvert x\rvert \to \infty$ unique

with $g \in C(\partial\Omega)$.

What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)

$$f(x) = O(g(x)) \Leftrightarrow \frac{f(x)}{g(x)} \le C \qquad f(x) = o(g(x)) \Leftrightarrow \frac{f(x)}{g(x)} \to 0$$

Uniqueness Proofs

$v=0$ on $\partial\Omega$

Dirichlet uniqueness: why?

Suppose I have u_1, u_2 with $\Delta u_{1,2}=0$ and $u_{1,2}=g$ $\quad v=u_1-u_2$

Neumann uniqueness: why?

Suppose you have $\tilde{u}=u_1-u_2$. Suppose $\tilde{u}$ is not a constant. Then $\nabla\tilde{u}\neq 0$ somewhere.

$$\int_\Omega \underbrace{\cancel{\tilde{u}\Delta\tilde{u}}}_{0}+\underbrace{\nabla\tilde{u}\cdot\nabla\tilde{u}}_{\|\nabla u\|^2}=\int_{\partial\Omega}\tilde{u}\,(\underbrace{\partial_n\tilde{u}}_{0})\,ds$$

$\Delta u_1=0 \quad \partial_n u_1=g$

$\Delta u_2=0 \quad \partial_n u_2=g$

Uniqueness: Remaining Points

Truth in advertising: Missing assumptions on Ω?

What's a DtN map?

Next mission: Find IE representations for each.