Today $\left(\frac{1}{2}I - D\right)$

- $N\left(\frac{I}{2} - D\right) = \{0\}$
- Existence + IEs for BVPs
- Corners
- $\Delta u = 0 \rightsquigarrow \Delta u + k^2 u = 0$

Laplace   Helmholtz

Announcements

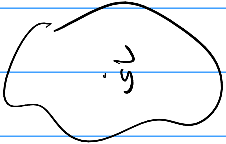
Project

HW 4

Green f. unbounded proof

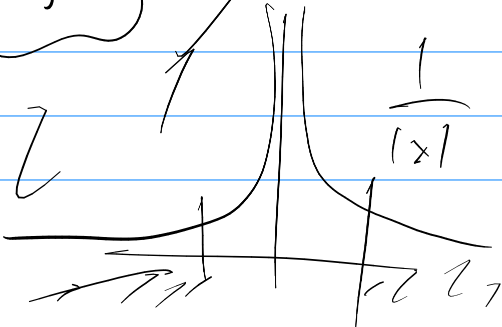
$\mathbb{R}^3$ $\quad u(x) = \frac{1}{|x-y|}$

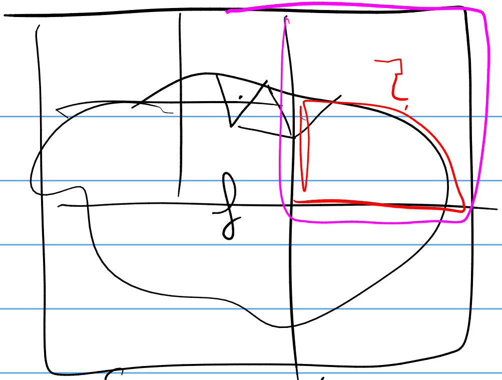
$\mathbb{R}^2 \to u(x) = \log|x-y|$



$$\frac{\partial}{\partial x_i} \underbrace{\frac{1}{\|x-y\|_2}}_{\Delta u = 0}$$

$\frac{1}{|x|}$





$$\int_{\Omega} G(x-y)\,\sigma(y)\,dy$$

Boundary Value Problems: Overview

	Dirichlet	**Neumann**
Int.	$\lim_{x\to\partial\Omega-} u(x) = g$ unique	$\lim_{x\to\partial\Omega-} \hat{n}\cdot\nabla u(x) = g$ may differ by constant
Ext.	$\lim_{x\to\partial\Omega+} u(x) = g$ $u(x) = \begin{cases} O(1) & 2D \\ o(1) & 3D \end{cases}$ as $\lvert x\rvert \to \infty$ unique	$\lim_{x\to\partial\Omega+} \hat{n}\cdot\nabla u(x) = g$ $u(x) = o(1)$ as $\lvert x\rvert \to \infty$ unique

with $g \in C(\partial\Omega)$.

What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)

$$\Delta u = 0 \quad \text{on } \Omega$$

$$\lim_{x \to \partial\Omega} u = g(x^*)$$

$\uparrow$ $x^* \in$

$$u(x) = D\sigma(x)$$

$$\cancel{u(x) = S_{\partial\Omega}\sigma(x)}$$

$$\cancel{\lim_{x \to x^* \in \partial\Omega^-} u(x) = g(x^*)}$$

$$\cancel{S\sigma(x^*) = g(x^*)}$$

~~(not second kind)~~

$$\lim_{x \to x^* \in \partial\Omega^-} u(x) = g(x^*)$$

$$\left(\tfrac{1}{2}I - D\right)\sigma = g(x^*)$$

Uniqueness Proofs

Dirichlet uniqueness: why?

Maximum principle $\Rightarrow$ Dirichlet uniqueness

Neumann uniqueness: why?

$$\left.\begin{array}{l}\Delta u = 0 \\ \partial_n u = g\end{array}\right) \qquad \nabla \bar{u} \neq 0$$

$$\int |\nabla u|^2 = 0$$

Uniqueness: Remaining Points

Truth in advertising: Missing assumptions on Ω?

C^2 domains

What's a DtN map?

Dirichlet-to-Neumann

Next mission: Find IE representations for each.

Uniqueness of Integral Equation Solutions

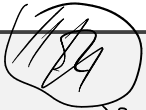
Theorem (Nullspaces [Kress LIE 2nd ed. Thm 6.20])

- $N(I/2 - D) = N(I/2 - S') = \{0\}$ ✓
- $N(I/2 + D) = \operatorname{span}\{1\}$, $N(I/2 + S') = \operatorname{span}\{\psi\}$, *where* $\int \psi \neq 0$.

IE Uniqueness: Proofs (I)

Show $N(I/2 - D) = \{0\}$.

Suppose $\frac{\varphi}{2} - D\varphi = 0$. To show: $\varphi = 0$



$\partial\Omega$

$u(x) := D\varphi$

$u^- = \frac{\varphi}{2} - D\varphi = 0$

$\Delta u = 0$

BVP unique $\rightarrow$ $u|_{\Omega} = 0$.

$(\partial_n u)^- = 0 \Rightarrow (\partial_n u)^+ = 0$

D' jump rel

$u|_{\Omega^c} = 0$

Because $[D\varphi] = \varphi$ and $[u] = 0 \Rightarrow \varphi = 0$.

IE Uniqueness: Proofs (II)

Show $N(I/2 - S') = \{0\}$.

Fredholm

IE Uniqueness: Proofs (III)

Show $N(I/2+D) = \operatorname{span}\{\cancel{1}\}$.

Suppose $\frac{\varphi}{2} + D\varphi = 0$. To show: φ const.

$u(x) := D\varphi$. $\quad u^+ = D\varphi + \varphi/2 = 0$

By ext. Dir. uniqueness: $u|_{\overline{\Omega}^c} = 0$. $\Rightarrow (\partial_n u)^+ = 0$

$\Rightarrow$ (D' jump rel.) $(\partial_n u)^- = 0 \Rightarrow$ int. Neumann, unique up to const.

$\Rightarrow u|_\Omega = c. \Rightarrow \varphi = [D\varphi] = \text{const.}$

What extra conditions on the RHS do we obtain?

int. Neumann:

$(\frac{I}{2} + S')(C(\partial\Omega)) = N(\frac{I}{2} + D)^\perp$

$= \operatorname{span}\{1\}^\perp$ ✓

ext. Dirichlet:

$(\frac{I}{2} + D)(C(\partial\Omega)) = N(\frac{1}{2} + S')^\perp$

$= \operatorname{span}\{\psi\}^\perp$

→ "Clean" Existence for 3 out of 4.

$(I - A)(X) = N(I - A^*)^\perp$

$$\Delta u = 0$$
$$(\partial_n u)^- = g$$
$$\int g = 0$$

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}\ldots$ do not know ψ. Use different kernel:

$$\hat{n} \cdot \nabla_y G(x,y) \qquad \to \qquad \hat{n} \cdot \nabla_y G(x,y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in \Omega$)

- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce $u = 0$ on exterior.
- $|x|^{n-2} u(x) = ?$ on exterior
- Thus $\int \phi = 0$. Contribution of the second term?
- $\phi/2 + D\phi = 0$, i.e. $\phi \in N(I/2 + D) = ?$
- Existence/uniqueness?

$\to$ Existence for 4 out of 4.