Today

- Choices in discr.
- Working in unstr. discr.
- Nyström

  $\|A_n - A\|_\infty \to 0$
- Anselone's Thm.

Announcements

- Presentation schedule
- Project proposals! 

Constructing Discrete Function Spaces

Floating point numbers (*Degrees of Freedom* or *DoFs*) $\leftrightarrow$ Functions

Discretization relies on three things:

- Base/reference domain
- Basis of functions
- Meaning of DoFs

modal
nodal

Related finite element concept: *Ciarlet triple*

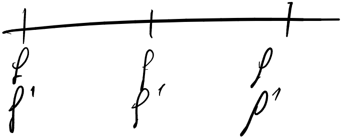
Discretization options for a curve?

What do the DoFs mean?

$$P_n \cdot P_m = \sum_{i=0}^{n+m} \alpha_i P_i$$

Common DoF choices:

- Point values of function
- Point values of (directional?) derivatives
- Basis coefficients
- Moments



Often: useful to have both "modes", "nodes", jump back and forth

modal

nodal

"spectral"

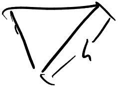
pseudospectral

Why high order?

$E \sim h$

Order p: Error bounded as $|u_h - u| \le Ch^p$

Thought experiment:



$\frac{1}{10} \sim h^5$

$E \sim h^5$

$h \sim 1.5x$

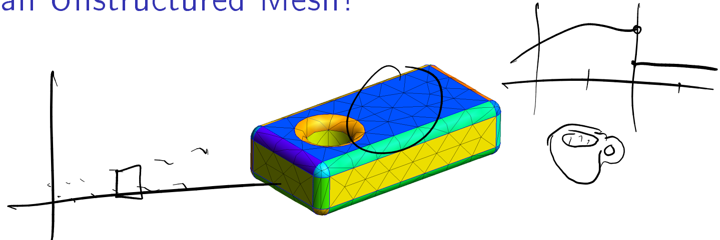
First order	Fifth order
1,000 DoFs ≈ 1,000 triangles	1,000 DoFs ≈ 66 triangles
rd. Error: 0.1	Error: 0.1
Error: 0.01 → ? 100,000	Error: 0.01 → ? 100 triangles

Complete the table.

Remarks:

- Want $p \geq 3$ available.
- Assumption: Solution sufficiently smooth
- Ideally: p chosen by user

What is an Unstructured Mesh?



Why have an unstructured mesh?

- deal with topology
- adaptive
- deal with nonsmoothness

What is the trade-off in going unstructured?

- complexity: data structures
- generated
- reference element

Demo: CAD software

Fixed-order vs Spectral

Fixed-order	Spectral
Number of DoFs n	Number of DoFs n
$\sim$	$\sim$
Number of 'elements'	Number of modes resolved
$\text{Error} \sim \frac{1}{n^p}$	$\text{Error} \sim \frac{1}{C^n}$
Examples?	Examples?
▶ Piecewise Polynomials	▶ Global Fourier
	▶ Global Orth. Polynomials

downside: full matrices
, structured discr.

What assumptions are buried in each of these?

sufficient smoothness

Fixed-order vs Spectral

What should the DoFs be?

What's the difficulty with purely modal discretizations?

Vandermonde Matrices

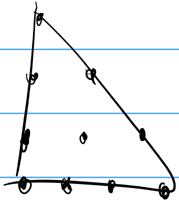
$$\begin{bmatrix} x_0^0 & x_0^1 & \cdots & x_0^n \\ x_1^0 & x_1^1 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n^0 & x_n^1 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = ?$$

Generalized Vandermonde Matrices

modal coeffs

nodal coefficients

$$\begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \cdots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \cdots & \phi_n(x_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \boxed{?}$$

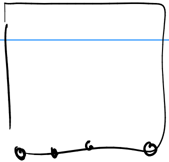


$P^3 \rightarrow$

$x^i y^j \quad i+j \le 3$

$0 \le i \le 3$

$0 \le j \le 3$



$Q^3 \quad x^i y^j$

Generalized Vandermonde Matrices

$$\begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \cdots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \cdots & \phi_n(x_n) \end{bmatrix} \text{MODAL COEFFS} = \text{NODAL COEFFS}$$

- Node placement? **Demo:** Interpolation node placement
- Vandermonde conditioning? **Demo:** Vandermonde conditioning
- What about multiple dimensions?
  - **Demo:** Visualizing the 2D PKDO Basis
  - **Demo:** 2D Interpolation Nodes

Common Operations

(Generalized) Vandermonde matrices simplify common operations:

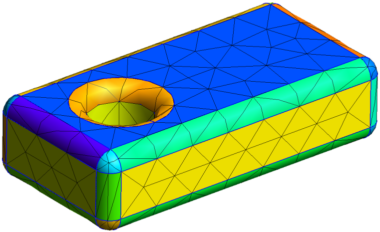
- Modal $\leftrightarrow$ Nodal ("Global interpolation")
  - Filtering
  - Up-/Oversampling
- Point interpolation (Hint: solve using V^T)
- Differentiation
- Indefinite Integration
- Inner product
- Definite integration

$p^3 \longrightarrow p^2$

$V_r V^{-1}$

$$\vec{\alpha}_f^T \vec{\alpha}_g = (V^{-1}\vec{f})^T (V^{-1}\vec{g})$$
$$= \vec{f}^T \underbrace{V^{-T} V^{-1}}_{M} \vec{g}$$

Unstructured Mesh



- Design a data structure to represent this [Nelements, dim, Nnodes]
- Compute normal vectors
- Compute area
- Compute integral of a function
- How is the function represented?

Integral Equation Discretizations: Overview

$\frac{e^{ikr}}{r}$?

$$\phi(x) - \int_\Gamma K(x,y)\phi(y)dy = f(y)$$

Nyström	Projection
- Approximate integral by quadrature: $\int_\Gamma f(y)dy \to \sum_{k=1}^{n} \omega_k f(y_k)$ - Evaluate quadrature'd IE at quadrature nodes, solve	- Consider residual: $R := \phi - A\phi - f$ - Pick projection P_n onto finite-dimensional subspace $P_n\phi := \sum_{k=1}^{n} \langle \phi, v_k \rangle w_k \quad \to$ DOFs $\langle \phi, v_k \rangle$ - Solve $P_n R = 0$

Projection/Galerkin

- Equivalent to projection: Test IE with test functions
- Important in projection methods: *sub*-space (e.g. of $C(\Gamma)$)

Name some generic discrete projection bases.

Polynomials $\longrightarrow$ Projection basis = density unknown $\rightarrow$ Galerkin

$\delta(x-y_i)$ $\neq$ $\rightarrow$ Petrov-Galerkin

Collocation and Nyström: the same?

collocation has to be exact.

Are projection methods implementable?

no.

Nyström Discretizations (I)

Nyström consists of two distinct steps:

1. Approximate integral by quadrature:

$x \in [\ \ldots\ ]$

$$\varphi_n(x) - \sum_{k=1}^{n} \omega_k K(x, y_k)\varphi_n(y_k) = f(x) \tag{1}$$

?

2. Evaluate quadrature'd IE at quadrature nodes, solve discrete system

$$\varphi_j^{(n)} - \sum_{k=1}^{n} \omega_k K(x_j, y_k)\varphi_k^{(n)} = f(x_j) \tag{2}$$

with $x_j = y_j$ and $\varphi_j^{(n)} = \varphi_n(x_j) = \varphi_n(y_j)$

Is version (1) solvable?

inf. rows , n columns ...?

Nyström Discretizations (II)

What's special about (2)?

Solution density also only known at point values. But: can get approximate continuous density. How?

Assuming the IE comes from a BVP. Do we also only get the BVP solution at discrete points?