Today

- Projection
  - ↳ Perturbation
  - ↳ function spaces
- Quadrature

Announcements

- Project Help
- Timeline for rest of class

Projection Method

X Banach space, $U \subset X$ nontrivial subspace, $A : X \to Y$ injective, $X_n \subset X$, $Y_n \subset Y$, $\dim X_n = n$, $\dim Y_n = n$, $P_n :? \to ?$

- P is a projection $\Leftrightarrow P|_U = \mathrm{Id} \Leftrightarrow P^2 = P$
- $\|P\| \geq 1$
- Orthogonal projectors: $\|P\| = 1$
- Interpolators ("collocation projection"): Also projections
- Projection method: $P_n A \phi_n = P_n f$ (#)

$A\varphi = f$

$\to \varphi_n = (P_n A)^{-1} P_n A \varphi$

Define convergence:

$\cancel{\varphi_n \to \varphi}$ $\quad \|\varphi_n - \varphi\|_\infty \to 0?$ $\quad (n \to \infty)$ $\quad \|\varphi_n - \varphi\| \leq C(n) \|\varphi\|$

Assumptions on the Approximation Spaces

What's needed of X_n so that it can even approximate the solution?

Denseness

$$\inf_{\psi \in X_n} \|\varphi - \psi\| \to 0 \qquad (n \to \infty)$$

Error Estimates for Projection

X Banach space, $A : X \to X$ injective, $P_n : Y \to Y_n$

Theorem (Céa's Lemma [Kress LIE 2nd ed. Thm 13.6])

Convergence of the projection method $\Leftrightarrow$ $\Leftarrow$
There exist n_0 *and* M *such that for* $n \geq n_0$

1. $P_n A : X_n \to Y_n$ *are invertible,*
2. $\|(P_n A)^{-1} P_n A\| \leq M$. *(Uniform Boundedness, Stability)*

In this case,

$$\|\phi_n - \phi\| \leq (1 + M) \inf_{\psi \in X_n} \|\phi - \psi\|$$

Céa's Lemma: Proof

Proof?

$$\varphi_n - \varphi = \left((P_n A)^{-1} P_n A - I\right)\varphi$$

$$\psi \in X_n: \left((P_n A)^{-1} P_n A - I\right)\psi = 0$$

$$\varphi_n - \varphi = \left((P_n A)^{-1} P_n A - I\right)(\varphi - \psi)$$

$$\|\varphi_n - \varphi\| \le \left(\|(P_n A)^{-1} P_n A\| + \|I\|\right)\|\varphi - \psi\|$$

Core message of the theorem?

Céa's Lemma: Remarks (1/2)

Note domain of invertibility for $P_n A$.

X_n

Domain/range of $(P_n A)^{-1} P_n A$?

Domain: X

Range: $(P_n A)^{-1} P_n A\, X$?

Céa's Lemma: Remarks (2/2)

$$P_n A \varphi_n = P_n f$$

$$P_n (A+B)\varphi_n = P_n f$$

Relationship to conditioning?

$$\|(P_n A)^{-1} P_n A\| \le \|(P_n A)^{-1}\| \, \|P_n A\| = \text{condnr.}$$

Relationship to second-kind?

$$A = (I - K)$$

Perturbations of Projection Methods

- $A : X \to X$ bounded linear operator
- with bounded inverse
- Projection method convergent for A (Céa)
- $B : X \to X$ bounded linear 'perturbation' operator

Theorem (Perturbations of projection methods [Kress LIE 2nd ed. Thm 13.7]**)**

If

- $\|P_n B|_{X_n}\| \to 0$ $(n \to 0)$, or
- *B compact and $A + B$ has no nullspace*

then the projection method still converges for $A + B$.

What is the compact perturbation B?

(could be) the result of applying quadrature/discrete comp.

Perturbation of Projections: Proof (1/2)

Prove part 2.

$$A(\underbrace{I+\underbrace{A^{-1}B}})=A+B$$

$I+A^{-1}B \Leftarrow$ Riesz/Fredholm: has inverse, $\|I+A^{-1}B\|<\infty$

A^{-1} bounded

$(P_nA)^{-1}(P_nA)\to I$ functionwise

$(P_nA)^{-1}P_n\to A^{-1}$ functionwise

$$\|(I+A^{-1}B)-(I+\underbrace{\underbrace{(P_nA)^{-1}P_n}_{\to A^{-1}\ \mathrm{pw}}B}_{\to A^{-1}B\ \text{norm}})\|\to 0$$

$[I+(P_nA)^{-1}P_nB]^{-1}$

$[I+\underbrace{\quad}_{?}B]^{-1}$

$\sup\|(L_n-L)A\|\to 0$

Perturbation of Projections: Proof (2/2)

'Discrete inverse' exists: $[I + (P_n A)^{-1} PnB]^{-1}$.

Set $S := A + B$. Then

$$P_n S = P_n A[I + (P_n A)^{-1} P_n B]$$

$$P_n(A+B)$$

is also invertible.

$$(P_n S)^{-1} P_n S = \underbrace{[I + (P_n A)^{-1} PnB]^{-1} (P_n A)^{-1}}_{(P_n S)^{-1}} \underbrace{P_n A(I + A^{-1} B)}_{P_n S}$$

So

$$\left\|(P_n S)^{-1} P_n S\right\| \leq \left\|[I + (P_n A)^{-1} PnB]^{-1}\right\| M \left\|I + A^{-1} B\right\|,$$

i.e. S has the stability property.