Today

- Rest
- Fredholm
- Spectral

Announcements

- HW 4 (Mat of top, BSS solve, O3x)
  - Project prop.
Weakly singular

\( G \subset \mathbb{R}^n \) compact

**Definition (Weakly singular kernel)**

- \( K \) defined, continuous everywhere except at \( x = y \)
- There exist \( C > 0, \alpha \in (0, n] \) such that

\[
|K(x, y)| \leq C|x - y|^\alpha - n \quad (x \neq y)
\]

**Theorem (Weakly singular kernel \( \Rightarrow \) compact [Kress LIE 2nd ed. Thm. 2.22])**

\( K \) weakly singular. Then

\[
(A\phi)(x) := \int_K K(x, y)\phi(y)\,dy.
\]

is compact on \( C(G) \).
Weakly singular: Proof Outline

Outline the proof of ‘Weakly singular kernel ⇒ compact’.
Weakly singular (on surfaces)

$\Omega \subset \mathbb{R}^n$ bounded, open, $\partial \Omega \in C'$

Definition (Weakly singular kernel (on a surface))

- $K$ defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n - 1]$ such that

$$|K(x, y)| \leq C|x - y|^{\alpha - n + 1} \quad (x, y \in \partial \Omega, \ x \neq y)$$

Theorem (Weakly singular kernel $\Rightarrow$ compact [Kress LIE 2nd ed. Thm. 2.23])

If $K$ weakly singular on $\partial \Omega$. Then $(A\phi)(x) := \int_{\partial \Omega} K(x, y)\phi(y)dy$ is compact on $C(\partial \Omega)$. 
Riesz Theory (I)

Still trying to solve

\[ L\phi := (I - A)\phi = \phi - A\phi = f \]

with \( A \) compact.

**Theorem (First Riesz Theorem [Kress, Thm. 3.1])**

\( N(L) \) is finite-dimensional.

Questions:
- What is \( N(L) \) again?
- Why is this good news?

\[ \forall \varphi \in N(L), L\varphi = 0 \]
\[ A\varphi = \varphi \]
\[ A |_{N(L)} = 1d \]
\[ \Rightarrow N(L) \text{ finite-dim} \]
Riesz First Theorem: Proof Outline

Show it.
Riesz Theory (II)

Theorem (Riesz theory [Kress, Thm. 3.4])

A compact. Then:

- $(I - A)$ injective $\iff$ $(I - A)$ surjective
- It's either bijective or neither s nor i.
- If $(I - A)$ is bijective, $(I - A)^{-1}$ is bounded.

Rephrase for solvability:

If $\forall \phi \in \mathcal{V} : \forall \phi : A\phi = \phi$ is solvable if $\ker(C) = \{0\} \iff \forall \alpha \phi : \forall \phi : A\phi = \phi$ only has the trivial solution.

Key shortcoming?

Goes away if $\ker(C) \neq \{0\}$. 
Suppose $C$ unbounded, then $\exists \varphi_n, \quad \|\varphi_n\| = 1, \quad \|C^{-1}\varphi_n\| > n$

$$g_n = \frac{D_n}{\|C^{-1}\varphi_n\|} \quad \varphi_n = \frac{C^{-1}\varphi_n}{\|C^{-1}\varphi_n\|}$$

$$\varphi_n - A \varphi_n = g_n \quad \varphi_n(x) : A \varphi_n(x) \to \varphi.$$
\[ \psi_n(u) - A \psi_n(u) = g_n(u) \rightarrow 0 \]

\[ \psi_n \rightarrow \psi \]

\[ \psi \in N(L) \implies (\psi - A\psi) = 0 \]

\[ \psi = 0 \] (assumed trivial nullspace, because L onto)
Hilbert spaces

Hilbert space: Banach space with a norm coming from an inner product:

\[(\alpha x + \beta y, z) =?\]

\[(x, \alpha y + \beta z) =?\]

\[\sqrt{(x, x)??} = \|x\|\]

\[(y, x) =?\]

Is \(C^0(G)\) a Hilbert space?

\(\|\cdot\|_\infty\)

Name a Hilbert space of functions.

\(L^2(\mathbb{R}) = \left\{ f : \int_\mathbb{R} |f|^2 \, dx < \infty \right\}\)
Continuous and Square-Integrable

Is \( C^0(G) \) “equivalent” to \( L^2(G) \)?

\[ \left| \langle \phi, \psi \rangle \right| \leq \| \psi \|_\infty \]

Why do compactness results transfer over nonetheless? Hint: What is

\[ \|(x, y)\| \leq \|x\| \cdot \|y\| \]
Adjoint Operators

Definition (Adjoint operator)

$A^*$ called adjoint to $A$ if

$$(Ax, y) = (x, A^*y)$$

for all $x, y$.

Facts:

- $A^*$ unique
- $A^*$ exists
- $A^*$ linear
- $A$ bounded $\Rightarrow$ $A^*$ bounded
- $A$ compact $\Rightarrow$ $A^*$ compact
Adjoint Operator: Observations?

What is the adjoint operator in finite dimensions? (in matrix representation)

\[ y_j = \sum A_{ij} x_i \]

What do you expect to happen with integral operators?

Swap targets and sources

\[ \Phi(x) = \int k(x, y) \phi(y) \, dy \]

Adjoint of the single-layer?

\[ S\sigma(x) = \int_\mathbb{R} \frac{1}{|x-y|} \sigma(y) \, dy \]

\[ S^* = S \]

Adjoint of the double-layer?

\[ D\sigma(x) = \int_\mathbb{R} \partial_{y_j} \frac{1}{|x-y|} \sigma(y) \, dy \]

\[ S'\sigma(x) = 2 \int_\mathbb{R} \frac{1}{|x-y|} \sigma(y) \, dy \]
Fredholm Alternative

**Theorem (Fredholm Alternative [Kress LIE 2nd ed. Thm. 4.14])**

\[ A : X \to X \text{ compact. Then either:} \]
- \[ I - A \text{ and } I - A^* \text{ are bijective} \leftarrow \]
  - or:
- \[ \dim N(I - A) = \dim N(I - A^*) \]
- \[ (I - A)(X) = N(I - A^*)^\perp \leftarrow \]
- \[ (I - A^*)(X) = N(I - A)^\perp \]

\[ (I - A)^* = I - A^* \]

Seen these statements before?

*Fundamental Thm. of Linear Algebra*
Fundamental Theorem of Linear Algebra

\[
\begin{align*}
\mathbb{R}^n & \rightarrow \mathbb{R}^m \\
\mathbb{R}^m & \rightarrow \mathbb{R}^n
\end{align*}
\]
Fredholm Alternative in IE terms

Translate to language of integral equation solvability:

If $\phi - \lambda \psi = 0 \Rightarrow \psi = 0 \Rightarrow \phi - \lambda \phi = 0$ has a unique solution.

\[
\phi(x) - \int k(x, y) \phi(y) \, dy = \phi(x) \quad \text{is solvable iff}
\]

$\phi + \phi$ for all solutions $\phi$ of

\[
\forall \phi \int k(\gamma, x) \phi(\gamma) \, d\gamma = 0.
\]

\[
(1 - A)(x) = N(1 - A^*)^+.
\]
Fredholm Alternative: Further Thoughts

What about symmetric kernels ($K(x, y) = K(y, x)$)?

$A = A^*$

Where to get uniqueness?

$\mathcal{I} - \mathcal{D} \psi = 0 \implies \psi$
Spectral Theory: Terminology

$A : X \to X$ bounded, $\lambda$ is a ... value:

**Definition (Eigenvalue)**

There exists an element $\phi \in X$, $\phi \neq 0$ with $A\phi = \lambda\phi$.

**Definition (Regular value)**

The “resolvent” $(\lambda I - A)^{-1}$ exists and is bounded.

Can a value be regular and “eigen” at the same time?

\[ A\psi - \lambda\psi = 0 \quad \text{no.} \]

What’s special about $\infty$-dim here?

Not all non-regular values are eigen.
Resolvent Set and Spectrum

**Definition (Resolvent set)**

\[ \rho(A) := \{ \lambda \text{ is regular} \} \]

**Definition (Spectrum)**

\[ \sigma(A) := \mathbb{C} \setminus \rho(A) \]
Spectral Theory of Compact Operators

\[
\begin{align*}
\sigma(A) & \neq \emptyset \\
\end{align*}
\]

**Theorem**

\( A : X \to X \) compact linear operator, \( X \) \( \infty \)-dim.

*Then:*

- \( 0 \in \sigma(A) \) (show!)
- \( \sigma(A) \setminus \{0\} \) consists only of eigenvalues
- \( \sigma(A) \setminus \{0\} \) is at most countable
- \( \sigma(A) \) has no accumulation point except for 0
Spectral Theory of Compact Operators: Proofs

Show first part.

Show second part.