Today
- \( \Delta u = 0 \) < Green's formula on unbounded domains
- Layer pol: jump relations
- BVP: uniqueness, existence

Announcements
- Projects
- HW4

\[ D \Psi (x) = \int_\Omega \frac{\partial}{\partial x_j} G(x,y) \frac{\partial \psi}{\partial x_j} \, dx \]
Mean Value Theorem

**Theorem (Mean Value Theorem [Kress LIE 2nd ed. Thm 6.7])**

If $\Delta u = 0$, $u(x) = \int_{B(x,r)} u(y) dy = \int_{\partial B(x,r)} u(y) dy$

Define $\mathcal{J}$?

$$\mathcal{J} = -\frac{1}{r} \int_{B(x,r)} B(x,y) \partial_y u(y) dy$$

Trace back to Green’s Formula (say, in 2D):

$$\int_{\partial \Omega} \partial_n u - \int_{\Omega} \Delta u = \frac{1}{2\pi} \int_{\partial \Omega} \partial_n \log r \int_{\Omega} u(y) dy ds_y$$
Maximum Principle

Theorem (Maximum Principle [Kress LIE 2nd ed. 6.9])

If $\triangle u = 0$ on compact set $\bar{\Omega}$:

- $u$ attains its maximum on the boundary.

Suppose it were to attain its maximum somewhere inside an open set. . .

that contradicts the mean value thm.

What do our constructed harmonic functions (layer potentials) do there?
Jump relations:

\[ S_{\mu}(\mathbf{x}) = \int_{\mathbf{y}} \frac{\log |\mathbf{x} - \mathbf{y}|}{2\pi} \mu(\mathbf{y} | \mathbf{y}) \]
Jump Relations: Mathematical Statement

Let \([X] = X_+ - X_-\). (Normal points towards “+” = “exterior”.)

**Theorem (Jump Relations [Kress LIE 2nd ed. Thm. 6.14, 6.17,6.18])**

\[
\begin{align*}
\lim_{x \to x_0^\pm} (S'\sigma) &= \left( S' \mp \frac{1}{2} I \right) (\sigma)(x_0) \quad \Rightarrow \quad [S'\sigma] = -\sigma \\
\lim_{x \to x_0^\pm} (D\sigma) &= \left( D \pm \frac{1}{2} I \right) (\sigma)(x_0) \quad \Rightarrow \quad [D\sigma] = \sigma
\end{align*}
\]

Truth in advertising: Assumptions on \(\Gamma\)?

\[C^2\]
Sketch the proof for the single layer.

uniformly conv. seq of cont. functions
Jump Relations: Proof Sketch for DLP

Sketch proof for the double layer.

\[ D \sigma(x) = \sigma(\varepsilon) \left( \frac{\partial A}{\partial x} \right)(x) + \int [\sigma - \sigma(\varepsilon)] \to 0 \]

\[ x \to \varepsilon + \]

x \to \Xi
Green’s Formula at Infinity: Motivation

$\Omega \subseteq \mathbb{R}^n$ bounded, $C^1$, connected boundary, $\triangle u = 0$ in $\mathbb{R}^n \setminus \Omega$, $u$ bounded

$$
(S_{\partial \Omega}(\hat{n} \cdot \nabla u) - D_{\partial \Omega}u)(x) + (S_{\partial B_r}(\hat{n} \cdot \nabla u) - D_{\partial B_r}u)(x) = u(x)
$$

for $x$ between $\partial \Omega$ and $B(x, r)$.

Behavior of individual terms? (20)
Green’s Formula at Infinity: Statement

\( u \) bounded, harmonic in \( \mathbb{R}^n \setminus \Omega \)

**Theorem (Green’s Formula in the exterior [Kress LIE 3rd ed. Thm 6.11])**

\[
(S_{\partial \Omega}(\hat{n} \cdot \nabla u) - D_{\partial \Omega} u)(x) + PV u_{\infty} = u(x)
\]

for some constant \( u_{\infty} \). Only for \( n = 2 \),

\[
u_{\infty} = \frac{1}{2\pi r} \int_{|y|=r} u(y) ds_y.
\]

Realize the power of this statement:

\( u, \partial u \) on all are potentially sufficient to represent \( u \).
Can we use this to bound $u$ as $x \to \infty$?
Consider the behavior of the fundamental solution as $r \to \infty$.

$$u(x) = u_\infty + O\left(\frac{1}{|x|}\right)$$

How about $u$'s derivatives?

$$\nabla u(x) = O\left(\frac{1}{|x|^{n-1}}\right)$$
Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

Laplace
Helmholtz
Calderón identities

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs
## Boundary Value Problems: Overview

### Dirichlet

<table>
<thead>
<tr>
<th>Int.</th>
<th>$\lim_{x \to \partial \Omega^-} u(x) = g$</th>
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<td>Ext.</td>
<td>$\lim_{x \to \partial \Omega^+} u(x) = g$</td>
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with $g \in C(\partial \Omega)$.

What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)

### Neumann

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$u(x) = o(1)$ as $|x| \to \infty$

+ unique
Uniqueness Proofs

Dirichlet uniqueness: why?
Suppose I have \( u_1, u_2 \) with \( \Delta u_1, u_2 = 0 \) and \( u_1 |_{\partial \Omega} = g, u_2 |_{\partial \Omega} = h \).

Neumann uniqueness: why?
Suppose you have \( \tilde{u} = u_1 - u_2 \). Suppose \( \tilde{u} \) is not a constant. Then \( \Delta \tilde{u} \neq 0 \) somewhere.

\[
\int_{\Omega} \Delta \tilde{u} + \nabla \tilde{u} \cdot \nabla \tilde{u} = \int_{\partial \Omega} \tilde{u} \left( \partial_n \tilde{u} \right) ds
\]

\( \Delta u_1 = 0 \quad \partial_n u_1 = g \)

\( \Delta u_2 = 0 \quad \partial_n u_2 = g \)
Uniqueness: Remaining Points

Truth in advertising: Missing assumptions on $\Omega$?

What’s a DtN map?

Next mission: Find IE representations for each.