Today

- unstructured mesh demos
- Nyström
- Collective compactness
- \( \| A - A_n \| \to 0 \)
- Andéon's theorem

Announcements

- Project
- HW4
Nyström Discretizations (I)

Nyström consists of two distinct steps:

1. Approximate integral by quadrature:

\[ \varphi_n(x) - \sum_{k=1}^{n} \omega_k K(x, y_k) \varphi_n(y_k) = f(x) \]  

(1)

2. Evaluate quadrature’d IE at quadrature nodes, solve discrete system

\[ \varphi_j^{(n)} - \sum_{k=1}^{n} \omega_k K(x_j, y_k) \varphi_k^{(n)} = f(x_j) \]  

(2)

with \( x_j = y_j \) and \( \varphi_j^{(n)} = \varphi_n(x_j) = \varphi_n(y_j) \)

Is version (1) solvable?

?  \( \begin{array}{c}
\text{full} \\
\text{skinny}
\end{array} \)
Nyström Discretizations (II)

What's special about (2)?

\[ \text{no continuous} \]

\emph{Solution} density also only known at point values. But: can get approximate continuous density. How?

\[ \tilde{\rho}_n(x) = \Psi(x) - \mathbb{E} \]

Assuming the IE comes from a BVP. Do we also only get the BVP solution at discrete points?

\[ n = 0 \quad \rightarrow \quad n = \sum \partial_n G(x-yj) \Psi(y) dy \]

\[ \begin{align*}
\nabla \cdot \omega &= f \\
\n\nabla \cdot \omega &= \sum \partial_n G(x-yj) \Psi(y_j) \omega_j
\end{align*} \]
Nyström Discretizations (III)

Does (1) ⇒ (2) hold?

Sure

Does (2) ⇒ (1) hold?

If you've chosen \( \Psi \) to be

the Nyström interpolant.
Nyström Discretizations (IV)

What good does that do us?

\[ \| \text{error} \|_\infty \quad \text{hard to eval far pt. values} \]

\[ \varphi - \varphi_n \]

\[ \therefore \text{we don't have to} \]

Does Nyström work for first-kind IEs?

\[ A \varphi = f \quad \text{no; no Nyström interp/la.} \]
Convergence for Nyström

Increase number of quadrature points $n$:
Get sequence $(A_n)$
Want $A_n \rightarrow A$ in some sense
What senses of convergence are there for sequences of functions $f_n$?

- uniform conv. (norm convergence) $\|f_n - f\| \rightarrow 0$
- pointwise $x : f_n(x) \rightarrow f(x)$

What senses of convergence are there for sequences of operators $A_n$?

- norm conv. $\|A - A_n\| \rightarrow 0$
- function wise $I : A \rightarrow A$ not strong enough
Convergence for Nyström (II) \[ A_n(\psi \psi_c) = 0 \]

Will we get norm convergence \[ \| A_n - A \|_\infty \to 0 \] for Nyström?

\[ \psi_c = 1 \text{ everywhere except in a } \varepsilon \text{-nbhd of the quad nodes} \quad \text{then it's } 0. \]

\[ \| A \psi \psi_c - A \psi \| \to 0 \max_{(x,y) \in (\varepsilon \to 0)} |S_h(x,y)/(\psi - 1)(x,y)| \]

Is functionwise convergence good enough?
\[ \| A - A_n \|_\infty \geq \| A \|_\infty \]

\[ \| A - A_n \| = \sup_{\| \eta \| = 1} \| (A - A_n) \eta \| \geq \| (A - A_n) \eta \psi_c \| \]

\[ = \| A \psi_c - A_n \psi_c \| \rightarrow 0 \]

\text{pf.}
Compactness-Based Convergence

\(X\) Banach space (think: of functions)

**Theorem (Not-quite-norm convergence [Kress LIE 2nd ed. Cor 10.4])**

\(A_n : X \rightarrow X\) bounded linear operators, functionwise convergent to \(A : X \rightarrow X\)
Then convergence is uniform on compact subsets \(U \subset X\), i.e.

\[
\sup_{\phi \in U} \|A_n \phi - A \phi\| \rightarrow 0 \quad (n \rightarrow \infty)
\]

How is this different from norm convergence? \(\|A_n\| \leq C\)

Only on a compact set
Collective Compactness

Set $\mathcal{A}$ of operators $A : X \to X$

**Definition (Collectively compact)**

$\mathcal{A}$ is called *collectively compact* if and only if for $U \subset X$ bounded, $\mathcal{A}(U)$ is relatively compact.

What was relative compactness (=precompactness)?

∃ conv. subseq.
Collective Compactness: Questions (1/2)

Is each operator in the set $\mathcal{A}$ compact?

Yes

Is collective compactness the same as “every operator in $\mathcal{A}$ is compact”?

No.
Collective Compactness: Questions (2/2)

When is a sequence collectively compact?

\[\text{view seq. as set}\]

\[\checkmark\]

Is the limit operator of such a sequence compact?

\[\checkmark\]

How can we use the two together?
Making use of Collective Compactness

$X$ Banach space, $A_n : X \to X$, $(A_n)$ collectively compact, $A_n \to A$ functionwise.

**Corollary (Post-compact convergence [Kress LIE 2nd ed. Cor 10.8])**

- $\|(A_n - A)A\| \to 0$
- $\|(A_n - A)A_n\| \to 0$
  ($n \to \infty$)
Anselone’s Theorem

\[(I - A) \phi = f\]

Assume:
\[(I - A)^{-1} \text{ exists, with } A : X \to X \text{ compact, } (A_n) : X \to X \text{ collectively compact and } A_n \to A \text{ functionwise.}\]

**Theorem (Nyström error estimate [Kress LIE 2nd ed. Thm 10.9])**

For sufficiently large \(n\), \((I - A_n)\) is invertible and

\[
\|\phi_n - \phi\| \leq C(\|(A_n - A)\phi\| + \|f_n - f\|)
\]

\[
C = \frac{1 + \|(I - A)^{-1}A_n\|}{1 - \|(I - A)^{-1}(A_n - A)A_n\|}
\]

\[
l + (I - A)^{-1}A = ?
\]

\[
1 + \frac{a}{1 - a} = \frac{l}{1 - a} + \frac{a}{1 - a} = \frac{1}{1 - a}
\]