November 3, 2026

Announcements

- HW5
- HW 4

Goals

- A-A
- Integral operators
  - ↳ continuous kernels
  - ↳ singular kernels
- Riesz

Review

Arzelà-Ascoli

Let $G \subset \mathbb{R}^n$ be compact.

Theorem (Arzelà-Ascoli [Kress LIE 3rd ed. Thm. 1.18])

$U \subset C(G)$ is precompact iff it is bounded and equicontinuous.

Equicontinuous means

$$\forall f \in U \quad \forall \varepsilon > 0 \ \exists \delta > 0 \ \text{s.t.} \ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Continuous means:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \text{s.t.} \ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Arzelà-Ascoli: Proof Sketch for b ∧ e ⇒ precompact

Let $\mathcal{U} \subset C(G)$ precompact + ec. Let $(f_n) \subset \mathcal{U}$.

Let (x_i) be a dense subseq in G. ($\mathcal{U}$ bounds.)

- Pick n_1 so that $(f_{n_1(k)}(x_1))_k$ is conv. ($(f_{n_1(k)})_k$)
- Pick subseq n_2 of n_1 so that $(f_{n_2(k)}(x_1))_k$ is conv.
- $(f_{n_i(k)}(x_j))_k$ converges for $j = 1, \ldots, i$
- $n(k) := n_k(k)$ "diagonal argument"

For each x_i $(f_{n(k)}(x_i))_k$ conv.

Arzelà-Ascoli: Proof Sketch for b ∧ e ⇒ precompact (cont'd)

Let $\varepsilon > 0$. Let $\delta > 0$ be what's needed for ec.

Consider the cover $(B(x_i, \delta))_i$. Covers all G.

$B(x_i, \delta)_{i=1}^{M}$ suffices to cover G.

$$|f_{n(k)}(x) - f_{n(\ell)}(x)| \le |f_{n(k)}(x) - f_{n(k)}(x_j)| \leftarrow < \varepsilon \text{ because of ec}$$

$$+ |f_{n(k)}(x_j) - f_{n(\ell)}(x_j)| \rightarrow \text{Cauchy + finite cover}$$

$$+ |f_{n(\ell)}(x_j) - f_{n(\ell)}(x)| \leftarrow$$

Arzelà-Ascoli (II)

Intuition?

"Uniformly continuous"?

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \quad |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

When does *uniform continuity* happen?

it if lives on a compact set.

(Note: Kress LIE 2nd ed. defines 'uniform equicontinuity' in one go.)

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis
- Norms and Operators
- Compactness
- Integral Operators
- Riesz and Fredholm
- A Tiny Bit of Spectral Theory

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

Integral Operators are Compact

$G \times G$

Theorem (Continuous kernel $\Rightarrow$ compact [Kress LIE 3rd ed. Thm. 2.28])

$G \subset \mathbb{R}^m$ compact, $K \in C(G^2)$. Then

$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

is compact on $C(G)$.

Use A-A. (a statement about compact *sets*) What is there to show?
Pick $U \subset C(G)$ bounded. $A(U)$ bounded?

Yes, we wrote down $\|A\|$

K uniformly cont on G!

Integral Operators are Compact (II)

Show that $A(U)$ is equicontinuous.

Let $\varepsilon > 0$. Use $\delta > 0$ from $U \subset\subset K$.
Let $\varphi \in U$. Let $\|x - y\| < \delta$.
To show.

$$|A\varphi(x) - A\varphi(y)| \leq \left| \int k(x,z)\varphi(z)\,dz - \int k(y,z)\varphi(z)\,dz \right|$$

$$\leq \int |k(x,z) - k(y,z)|\,|\varphi(z)|\,dz$$

(k is uniformly continuous)

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n]$ such that

$$|K(x,y)| \leq C|x-y|^{\alpha-n} \qquad (x \neq y)$$

Theorem (Weakly singular kernel $\Rightarrow$ compact [Kress LIE 3rd ed. Thm. 2.29])

K weakly singular. Then

$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

is compact on $C(G)$, where $\mathrm{cl}(G^\circ) = G$.

Weakly singular: Proof Outline

Outline the proof of 'Weakly singular kernel $\Rightarrow$ compact'.

$$\int_{B(0,d)} |y|^{\alpha-n} = \omega_n \int \rho^{\alpha-n} \rho^{n-1} d\rho < \infty$$

Weakly singular (on surfaces)

$\Omega \subset \mathbb{R}^n$ bounded, open, $\partial\Omega$ is C^1 (what does that mean?)

Definition (Weakly singular kernel (on a surface))

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n-1]$ such that

$$|K(x,y)| \le C|x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

Theorem (Weakly singular kernel $\Rightarrow$ compact [Kress LIE 3rd ed. Thm. 2.30])

K weakly singular on $\partial\Omega$. Then $(A\phi)(x) := \int_{\partial\Omega} K(x,y)\phi(y)dy$ is compact on $C(\partial\Omega)$.

Q: Has this estimate gotten worse or better?

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Riesz Theory (I)

Still trying to solve

$$L\phi := (I - A)\phi = \phi - A\phi = f$$

with A compact.

Theorem (First Riesz Theorem [Kress LIE 3rd ed., Thm. 3.1])

$N(L)$ is finite-dimensional.

Questions:

- What is $N(L)$ again?
- Why is this good news?

Riesz First Theorem: Proof Outline

Show it.

- $N(L)$ is closed. (continuity of L)
- $L\varphi = 0 \Rightarrow L\varphi = (I - A)\varphi = \varphi - A\varphi \qquad \varphi = A\varphi$
- $A : N(L) \to N(L)$ compact.
- (identity compact only on finite-dim subs.

Riesz Theory (II)

Theorem (Riesz theory [Kress LIE 3rd ed., Thm. 3.4])

A compact. Then:

- $(I - A)$ *injective* $\Leftrightarrow$ $(I - A)$ *surjective*
  - *It's either bijective or neither s nor i.*
- *If* $(I - A)$ *is bijective,* $(I - A)^{-1}$ *is bounded.*

Rephrase for solvability:

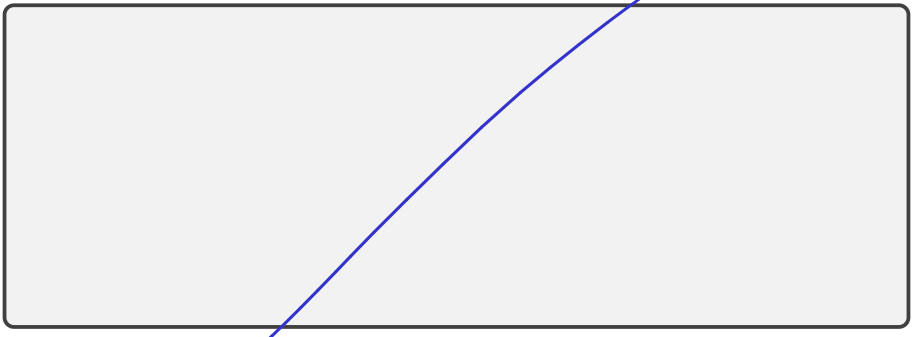
$(I-A)\varphi=0$ uniquely $\psi=0 \Rightarrow (I-A)\varphi=\rho$ has a unique sol.

Key shortcoming?

gives onL if nullspace.

Riesz Theory: Boundedness Proof Outline

Assuming $(I - A)$ is bijective, show that $(I - A)^{-1}$ is bounded.



(Quite a bit more to the proof: Riesz' second, third theorems, Riesz numbers, ...)

Hilbert spaces

Hilbert space: Banach space with a norm coming from an *inner product*:

$$(\alpha x + \beta y, z) = ?$$

$$(x, \alpha y + \beta z) = ?$$

$$(x, x)?$$

$$(y, x) = ?$$

Is $C^0(G)$ a Hilbert space?

no

Name a Hilbert space of functions.

$L^2(\Omega) := \{ \|f\|_{L^2(\Omega)} < \infty \}$

$(f, g) = \int_\Omega f \cdot g$ $\qquad \|f\|_{L^2} = \sqrt{(f, f)}$

Continuous and Square-Integrable

Can we carry over $C^0(G)$ boundedness/compactness results to $L^2(G)$?

X, Y normed spaces with a scalar product so that $|(\phi, \psi)| \leq \|\phi\| \, \|\psi\|$ for $\phi, \psi \in X$.

Theorem (Lax dual system [Kress LIE 3rd ed. Thm. 4.13])

Let $U \subseteq X$ be a subspace and let $A : X \to Y$ and $B : Y \to X$ be bounded linear operators with

$$(A\phi, \psi) = (\phi, B\psi) \qquad (\phi \in U, \psi \in Y).$$

Then $A : U \to Y$ is bounded with respect to $\|\cdot\|_s$ induced by the scalar product and $\|A\|_s^2 \leq \|A\| \, \|B\|$.

Based on this, it is also possible to carry over compactness results.