November 5, 2026

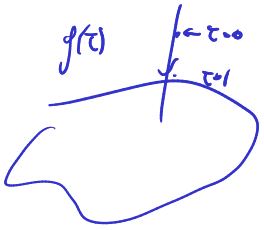
Announcements

- HW4 due Mon
- HW5 out
- HW2 graded

Goals

- Adjoints
- Fredholm
- Spectral theory
- $\Delta u = 0$

Review



$$(\mathcal{D}\sigma)(x) = \int \partial_{n_y} G(x-y)\,\sigma(y)\,ds_y$$

$$\sum_j \partial_{n_{y_j}} G(x-y_j)\,\sigma(y_j)\,\omega_j$$

Continuous and Square-Integrable

Can we carry over $C^0(G)$ boundedness/compactness results to $L^2(G)$?

X, Y normed spaces with a scalar product so that $|(\phi, \psi)| \leq \|\phi\| \, \|\psi\|$ for $\phi, \psi \in X$.

Theorem (Lax dual system [Kress LIE 3rd ed. Thm. 4.13])

Let $U \subseteq X$ be a subspace and let $A : X \to Y$ and $B : Y \to X$ be bounded linear operators with

$$(A\phi, \psi) = (\phi, B\psi) \qquad (\phi \in U, \psi \in Y).$$

Then $A : U \to Y$ is bounded with respect to $\|\cdot\|_s$ induced by the scalar product and $\|A\|_s^2 \leq \|A\| \, \|B\|$.

Based on this, it is also possible to carry over compactness results.

Adjoint Operators

Definition (Adjoint oeprator)

A^* called adjoint to A if

$$(Af, g) = (f, A^*g)$$

for all x, y.

Facts:

- A^* unique
- A^* exists
- A^* linear
- A bounded $\Rightarrow$ A^* bounded
- A compact $\Rightarrow$ A^* compact

Adjoint Operator: Observations?

What is the adjoint operator in finite dimensions? (in matrix representation)

$.^H$ $.^T$

What do you expect to happen with integral operators?

swap source $\longleftrightarrow$ target

Adjoint of the single-layer?

S $\int G(x-y)\,\sigma(y)$

Adjoint of the double-layer?

S' $\int \partial_{n_y} G(x-y)\,\sigma(y)\,dS_y$

Fredholm Alternative

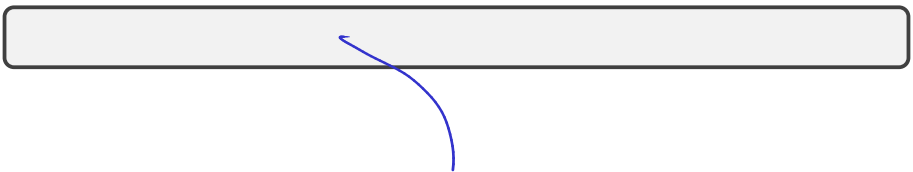
Theorem (Fredholm Alternative [Kress LIE 3rd ed. Thm. 4.17])

$A : X \to X$ compact. Then either:

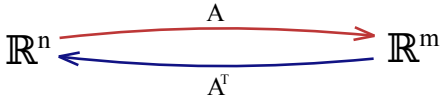
- *$I - A$ and $I - A^*$ are bijective*

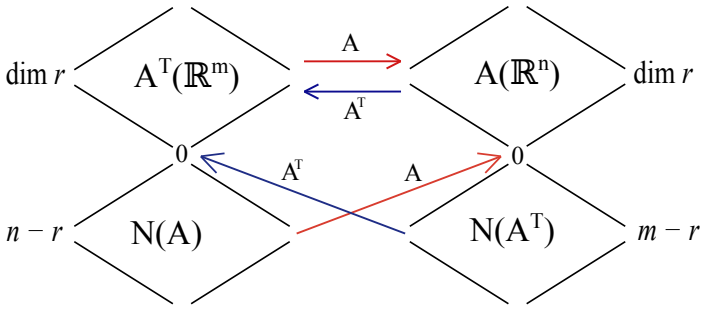
  or:
- $\dim N(I - A) = \dim N(I - A^*)$
- $(I - A)(X) = N(I - A^*)^\perp$
- $(I - A^*)(X) = N(I - A)^\perp$

Seen these statements before?



Fundamental Theorem of Linear Algebra





[Credit: Wikipedia]

Fredholm Alternative in IE terms

Translate to language of integral equation solvability:

- $(I - A)(X) = N(I - A^*)^\perp$
- $(I - A^*)(X) = N(I - A)^\perp$

Either $\varphi(x) - \int K(x,y)\,\varphi(x)\,dx = 0$ and

$\psi(x) - \int K(y,x)\,\psi(y)\,dx = 0$ only have trivial sol; and

or hom. and inhom. IE have the same dimension of solution spaces — everybody solvable

$\varphi(x) - \int K(x,y)\,\varphi(x)\,dx = f(x)$

$\psi(x) - \int K(y,x)\,\psi(y)\,dx = g(x)$

are solvable iff $f \perp \psi$ for all solutions $(I - A^*)\psi = 0$

Fredholm Alternative: Further Thoughts

What about symmetric kernels ($K(x, y) = K(y, x)$)?

Where to get uniqueness?

do that (almost) next.

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis
- Norms and Operators
- Compactness
- Integral Operators
- Riesz and Fredholm
- A Tiny Bit of Spectral Theory

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

Spectral Theory: Terminology

$A : X \to X$ bounded, λ is a ... value:

Definition (Eigenvalue)

There exists an element $\phi \in X$, $\phi \neq 0$ with $A\phi = \lambda\phi$.

Definition (Regular value)

The "resolvent" $(\lambda I - A)^{-1}$ exists and is bounded.

Can a value be regular and "eigen" at the same time?

eigen : $N(\lambda I - A) \neq \{0\}$ $\Rightarrow$ not regular

What's special about ∞-dim here?

Not all non-regular are eigen.

Resolvent Set and Spectrum

$$\lambda I - A = 0$$
$$I - \frac{A}{\lambda} = \frac{1}{\lambda}$$

Definition (Resolvent set)

$\rho(A) := \{\lambda \text{ is regular}\}$

Definition (Spectrum)

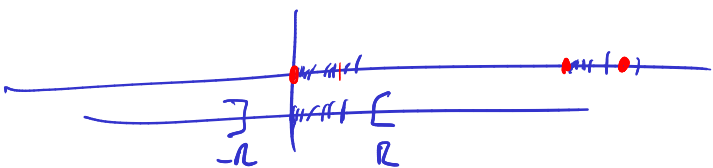
$\sigma(A) := \mathbb{C} \setminus \rho(A)$

Spectral Theory of Compact Operators

Theorem

$A : X \to X$ compact linear operator, X ∞-dim.
Then:

- $0 \in \sigma(A)$
- $\sigma(A) \setminus \{0\}$ *consists only of eigenvalues*
- $\sigma(A) \setminus \{0\}$ *is at most countable*
- $\sigma(A)$ *has no accumulation point except for 0*



Spectral Theory of Compact Operators: Proofs

Show the first part.

Let $0 \notin \sigma(A)$. Then A^{-1} invertible. $I = A A^{-1}$ (compact A, compact)

$\Rightarrow$ only in finite dim

Show second part.

Riesz

Spectral Theory of Compact Operators: Implications

Rephrase last two: how many eigenvalues with $|\cdot| \geq R$?

Recap: What do compact operators do to high-frequency data?

damper

Don't confuse $I - A$ with A itself!

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory
- Singular Integrals
- Green's Formula and Its Consequences
- Jump Relations

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory
- Singular Integrals
- Green's Formula and Its Consequences
- Jump Relations

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

Recap: Layer potentials

$$(S\sigma)(x) := \int_\Gamma G(x-y)\sigma(y)ds_y$$

$$(S'\sigma)(x) := PV\ \hat{n}\cdot\nabla_x \int_\Gamma G(x-y)\sigma(y)ds_y$$

$$(D\sigma)(x) := PV \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\sigma(y)ds_y$$

$$\rightarrow\ (D'\sigma)(x) := f.p.\ \hat{n}\cdot\nabla_x \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\sigma(y)ds_y$$

Definition (Harmonic function)

$\triangle u = 0$

Where are layer potentials harmonic?

Away from Γ.

On the double layer again

Is the double layer *actually* weakly singular? Recap:

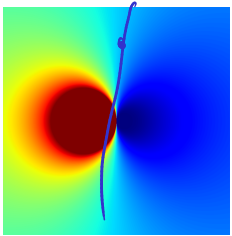
Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n-1]$ such that

$$|K(x,y)| \leq C|x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

Actual Singularity in the Double Layer (2D)

$$\frac{\partial}{\partial_x} \log(|0 - x|) = \frac{x}{x^2 + y^2}$$



$n_y \cdot (x - y) = O(|x-y|^2)$

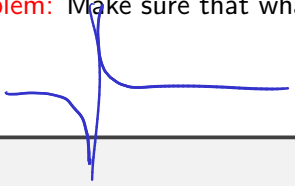
holds for smooth surfaces

- Singularity with approach on $y = 0$?
- Singularity with approach on $x = 0$?

Cauchy Principal Value

But I don't want to integrate across a singularity! $\rightarrow$ punch it out.

Problem: Make sure that what's left over is well-defined

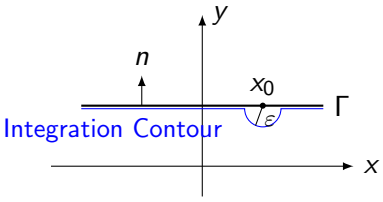


$$\int_{-1}^{1} \frac{1}{x} dx?$$

$$\int_0^1 \frac{1}{x} dx$$

$$PV \int_{-1}^{1} \frac{1}{x} := \lim_{\varepsilon \to 0+} \left(\int_{-1}^{-\varepsilon} \frac{1}{x} dx + \int_{\varepsilon}^{1} \frac{1}{x} dx \right)$$

Principal Value in n dimensions



Again: Symmetry matters!

What about even worse singularities?

Hadamard Finite part.

Recap: Layer potentials

$$(S\sigma)(x) := \int_\Gamma G(x-y)\sigma(y)ds_y$$

$$(S'\sigma)(x) := \text{PV } \hat{n}\cdot\nabla_x \int_\Gamma G(x-y)\sigma(y)ds_y$$

$$(D\sigma)(x) := \text{PV} \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\sigma(y)ds_y$$

$$(D'\sigma)(x) := f.p.\ \hat{n}\cdot\nabla_x \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\sigma(y)ds_y$$

Important for us: Recover 'average' of interior and exterior limit without having to refer to off-surface values.

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory
- Singular Integrals
- Green's Formula and Its Consequences
- Jump Relations

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

Green's Theorem

Ω bounded

Theorem (Green's Theorem [Kress LIE 2nd ed. Thm 6.3])

$$\int_\Omega u\triangle v + \nabla u \cdot \nabla v = \int_{\partial\Omega} u(\hat{n}\cdot\nabla v)ds$$

$$\longrightarrow \int_\Omega u\triangle v - v\triangle u = \int_{\partial\Omega} u(\hat{n}\cdot\nabla v) - v(\hat{n}\cdot\nabla u)ds$$

If $\triangle v = 0$ and $u = 1$, then

$$\int_{\partial\Omega} \hat{n}\cdot\nabla v = ?$$

$0 \int u\triangle v - \int v\triangle u = \int \partial_n v$

Green's Formula

$-\Delta G = \delta$

What if $\triangle v = 0$ and $u = G(|y - x|)$ in Green's second identity?

$$\int_\Omega u\triangle v - v\triangle u = \int_{\partial\Omega} u(\hat{n}\cdot\nabla v) - v(\hat{n}\cdot\nabla u)ds$$

0

Can you write that more briefly?

$+\, v(x) = +\int v\delta = S(\partial_n v) - Dv$

Green's Formula (Full Version)

Ω bounded

Theorem (Green's Formula [Kress LIE 2nd ed. Thm 6.5])

If $\triangle u = 0$, *then*

$$(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} u(x) & x \in \Omega, \\ \frac{u(x)}{2} & x \in \partial\Omega, \\ 0 & x \notin \Omega. \end{cases}$$

$-D1 = \begin{cases} 1 \\ \frac{1}{2} \\ 0 \end{cases}$

Green's Formula and Cauchy Data

Suppose I know 'Cauchy data' $(u|_{\partial\Omega},\ \hat{n}\cdot\nabla u|_{\partial\Omega})$ of u. What can I do?

eval u anywhere

What if Ω is an exterior domain?

What if $u = 1$? Do you see any practical uses of this?