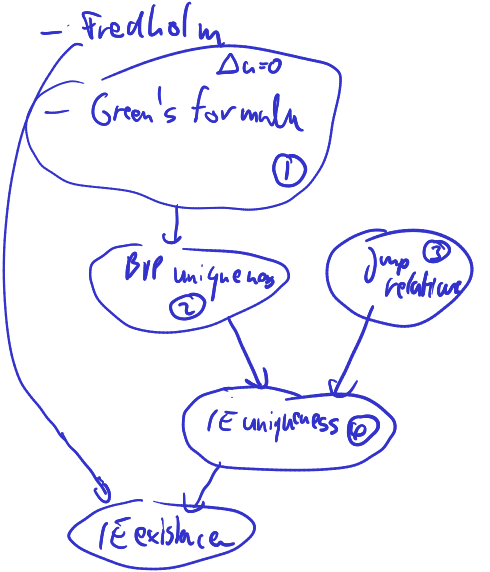
November 10, 2026

Announcements

- HW5

Goals

Review



Green's Formula (Full Version)

Ω bounded

Theorem (Green's Formula [Kress LIE 2nd ed. Thm 6.5])

If $\triangle u = 0$, *then*

$$(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} u(x) & x \in \Omega, \\ \frac{u(x)}{2} & x \in \partial\Omega, \\ 0 & x \notin \Omega. \end{cases}$$

Green's Formula and Cauchy Data

Suppose I know 'Cauchy data' $(u|_{\partial\Omega},\ \hat{n}\cdot\nabla u|_{\partial\Omega})$ of u. What can I do?

What if Ω is an exterior domain?

later

What if $u = 1$? Do you see any practical uses of this?



Mean Value Theorem

Theorem (Mean Value Theorem [Kress LIE 2nd ed. Thm 6.7])

If $\Delta u = 0$, $u(x) = \overline{\int}_{B(x,r)} u(y)dy = \overline{\int}_{\partial B(x,r)} u(y)dy$

Define $\overline{\int}$?

$\int \partial_n u = 0$

$|\Omega| := \int_\Omega 1 \, d\vec{x}$ $\qquad \overline{\int} \rho(x) \, dx = \frac{1}{|\Omega|} \int \rho(\vec{x}) \, d\vec{x}$

Trace back to Green's Formula (say, in 2D):

$x \in B(0,r)$: $\partial B(0,r)$ $\quad u(x) = S\partial_n u - D u = \int_{\partial B(0,r)} K(x,y) \, \partial_n u + \dots$

$= \frac{1}{2\pi} \log r \int \partial_n u - \frac{1}{2\pi r} \int_{\partial B} u$ ($\int \partial_n u = 0$)

Maximum Principle

$\partial_x^2 u + \partial_y^2 u = 0$

Theorem (Maximum Principle [Kress LIE 2nd ed. 6.9])

If $\triangle u = 0$ on compact set $\bar{\Omega}$:
u attains its maximum on the boundary.

Suppose it were to attain its maximum somewhere inside an open set...

contradicts mean value

Green's Formula at Infinity: Statement

$\Omega \subseteq \mathbb{R}^n$ bounded, C^1, connected boundary, $\triangle u = 0$ in $\mathbb{R}^n \setminus \Omega$, u bounded

Theorem (Green's Formula in the exterior [Kress LIE 3rd ed. Thm 6.11]**)**

$$(-S_{\partial\Omega}(\hat{n} \cdot \nabla u) + D_{\partial\Omega} u)(x) + \mathrm{PV} u_\infty = u(x)$$

for some constant u_∞. *Only for* $n = 2$,

$$u_\infty = \frac{1}{2\pi r} \int_{|y|=r} u(y) ds_y.$$



Realize the power of this statement:

Far field is up to the kernels!

Green's Formula at Infinity: Proof (1/4)

We will focus on $\mathbb{R}^3$. WLOG assume $0 \in \Omega$. Let $M = \|u\|_{L^\infty(\mathbb{R}^n \setminus \bar{\Omega})}$.
First, show $\|\nabla u\| \leq 6M/\|x\|$ for $x \geq R_0$.

Green's Formula at Infinity: Proof (2/4)

Let $x \in \mathbb{R}^3 \setminus \bar{\Omega}$. Let r be such that $\bar{\Omega} \subset B(x, r)$. Apply Green's formula on *bounded* domains to $B(x, r) \setminus \bar{\Omega}$:

$$(S_{\partial\Omega}(\partial_n u) - D_{\partial\Omega} u)(x) + (S_{\partial B(x,r)}(\partial_n u) - D_{\partial B(x,r)} u)(x) = u(x).$$

Show $S_{\partial B(x,r)}(\partial_n u) \to 0$ as $r \to \infty$:

Green's Formula at Infinity: Proof (3/4)

It remains to figure out the term

$$(D_{\partial B(x,r)}u)(x) = \frac{4\pi}{r^2}\int_{\partial B(x,r)} u(y)dS_y.$$

Can we transplant that ball to the origin in some sense?



Green's Formula at Infinity: Proof (4/4)

Observe

$$\left| \frac{4\pi}{r^2} \int_{\partial B(0,r)} u(y) dS_y \right| \leq 4\pi M.$$

Consider the sequence

$$\mu_n := \frac{4\pi}{r_n^2} \int_{\partial B(0,r_n)} u(y) dS_y.$$

Because of its boundedness and sequential compactness of the bounding interval, out of a sequence of radii r_n, we can pick a subsequence so that $(\mu_{n(k)})$ converges. Call the limit u_∞.

Green's Formula at Infinity: Impact

Can we use this to bound u as $x \to \infty$?
Consider the behavior of the kernel as $r \to \infty$. Focus on 3D for simplicity. (But 2D holds also.)

$$u(x) = u_\infty + O\left(\frac{1}{\|x\|}\right) \quad \text{as } \|x\| \to \infty$$

How about u's derivatives?

$$Du = O\left(\frac{1}{\|x\|^{n-1}}\right) = \begin{cases} \frac{1}{r} & 2D \\ \frac{1}{r^2} & 3D \end{cases}$$

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory
- Singular Integrals
- Green's Formula and Its Consequences
- Jump Relations

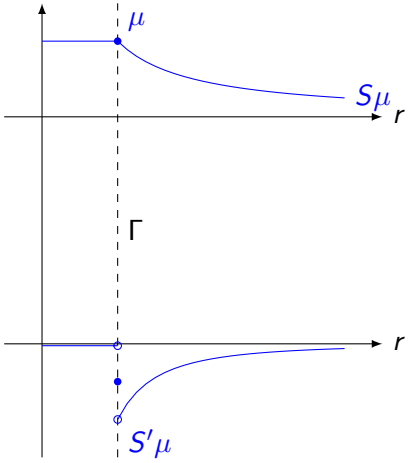
Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

Jump relations:



Jump Relations: Mathematical Statement

Let $[X] = X_+ - X_-$. (Normal points towards "+"="exterior".) Let $x_0 \in \Gamma$.

Theorem (Jump Relations [Kress LIE 3rd ed. Thm. 6.15, 6.18,6.19])

$$[S\sigma] = 0$$

$$\lim_{x \to x_0 \pm} (S'\sigma) = \left(S' \mp \frac{1}{2} I \right)(\sigma)(x_0) \quad \Rightarrow \quad [S'\sigma] = -\sigma$$

$$\lim_{x \to x_0 \pm} (D\sigma) = \left(D \pm \frac{1}{2} I \right)(\sigma)(x_0) \quad \Rightarrow \quad [D\sigma] = \sigma$$

$$[D'\sigma] = 0$$

Truth in advertising: Assumptions on Γ?

C^2 boundaries

Jump Relations: Proof Sketch for SLP

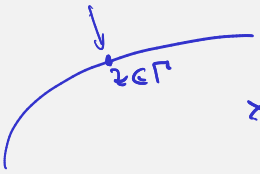
Sketch the proof for the single layer.

analogous to weak singularity proof

Jump Relations: Proof Sketch for DLP

$D(\alpha f) = \alpha Df$

Sketch proof for the double layer.



$x = z + h\hat{n}(z)$

$$D\sigma(x) - D[\sigma(z)](x) + D\sigma(x) - D[\sigma(z)](x)$$

$$\sigma(z)(D1)(x) + D[\sigma - \sigma(z)](x)$$

$\to 0$

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Boundary Value Problems: Overview

	Dirichlet	**Neumann**
Int.	$\lim_{x\to\partial\Omega-} u(x) = g$ ⊕ unique	$\lim_{x\to\partial\Omega-} \hat{n}\cdot\nabla u(x) = g$ ⊖ may differ by constant
Ext.	$\lim_{x\to\partial\Omega+} u(x) = g$ $u(x) = \begin{cases} O(1) & 2D \\ o(1) & 3D \end{cases}$ as $\lvert x\rvert \to \infty$ ⊕ unique	$\lim_{x\to\partial\Omega+} \hat{n}\cdot\nabla u(x) = g$ $u(x) = o(1)$ as $\lvert x\rvert \to \infty$ ⊕ unique

with $g \in C(\partial\Omega)$.

What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)

$f(x) = O(g(x))$ $\quad \frac{f(x)}{g(x)} \leq C$

$f(x) = o(g(x))$ $\quad \frac{f(x)}{g(x)} \to 0$

Uniqueness Proofs

Dirichlet uniqueness: why?

maximum principle

Neumann uniqueness: why?

Suppose u_1, u_2 both solve the BVP, $\tilde{u} = u_1 - u_2$

$\tilde{u}$ not constant. $\nabla \tilde{u} \neq 0$ somewhere.

$$\int u \Delta u + \nabla u \cdot \nabla u = \int_{\partial\Omega} u (\partial_n u)$$

Apply to $\tilde{u}$

$$\int \tilde{u} \Delta \tilde{u} + \nabla \tilde{u} \cdot \nabla \tilde{u} = \int_{\partial\Omega} \tilde{u} (\partial_n \tilde{u})$$

$$\int |\nabla \tilde{u}|^2 = 0$$

Uniqueness: Remaining Points

Truth in advertising: Missing assumptions on Ω?



What's a DtN map?

Finding IE representations

$\lim_{x \to x_0 t} D\sigma(x) = \left(D^{on}\sigma \pm \frac{\sigma}{2} \right)(x_0)$

Find integral representations that lead to second-kind IEs for each of the BVPs:

	Dirichlet	**Neumann**
Int.	$u(x) := D\sigma(x)$ $-\frac{\sigma}{2} + D^{on}\sigma = g$	$u(x) := S\sigma(x)$ $\frac{\sigma}{2} + S'\sigma = g$
Ext.	$u(x) := D\sigma(x)$ $\frac{\sigma}{2} + D^{on} = g$	$u(x) := -S\sigma(x)$ $-\frac{\sigma}{2} + S'\sigma = g$

Uniqueness of Integral Equation Solutions

Theorem (Nullspaces [Kress LIE 3rd ed. Thm 6.21])

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \mathrm{span}\{1\}$, $N(I/2 + S') = \mathrm{span}\{\psi\}$, *where* $\int \psi \neq 0$.

Disclaimer: $\int \psi \neq 0$ not shown here, takes a little extra work.

IE Uniqueness: Proofs (1/3)

Show $N(I/2 - D) = \{0\}$.

Suppose $\varphi_2 - D\varphi = 0$. To show. $\varphi = 0$.

Define $u(x) := D\sigma(x)$ $\quad \lim_{x \to \partial\Omega^-} u(x) = D\varphi \cdot \frac{\varphi}{2} = 0$

$\Rightarrow u|_\Omega = 0.$

$\Rightarrow \partial_n u(+) = 0$ $\Rightarrow$ BVP uniqueness $\Rightarrow u(ext) = 0$

$0 = [D\varphi] = \varphi$