November 12, 2026

Announcements

- HW5
- Project talk date assignments
- Project due date.

Goals

- IE existence for "complicated" case
  - ↳ non-smooth boundaries
- Helmholtz
- Calderón

Review

Finding IE representations

Find integral representations that lead to second-kind IEs for each of the BVPs:

	Dirichlet	**Neumann**
Int.	$u(x) := D\sigma$ $-\frac{\sigma}{2} + D\sigma = g$	$u(x) := S\sigma$ $\frac{\sigma}{2} + S'\sigma = g$
Ext.	$u(x) := D\sigma$ $\frac{\sigma}{2} + D\sigma = g$	$-\frac{\sigma}{2} + S'\sigma = g$

Uniqueness of Integral Equation Solutions

$(I - A)(x) = N(I - A^*)^{\perp}$ $\int_\Omega \partial_n u = 0$

Theorem (Nullspaces [Kress LIE 3rd ed. Thm 6.21])

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \mathrm{span}\{1\}$, $N(I/2 + S') = \mathrm{span}\{\psi\}$, *where* $\int \psi \neq 0$.

Disclaimer: $\int \psi \neq 0$ not shown here, takes a little extra work.

IE Uniqueness: Proofs (1/3)

Show $N(I/2 - D) = \{0\}$.

last time

IE Uniqueness: Proofs (2/3)

Show $N(I/2 - S') = \{0\}$.

Fredholm (adjoint,

IE Uniqueness: Proofs (3/3)

Show $N(I/2 + D) = \text{span}\{1\}$.

"$\subseteq$": Suppose $\frac{\varphi}{2} + D\varphi = 0$. To show: $\varphi = \text{const}$.

Let $u(x) := \mathcal{D}\varphi(x)$. $\Rightarrow u^+ = \lim_{x \to \partial\Omega^+} u(x) = 0$

$\Rightarrow u \equiv 0$ on the ext. $\Rightarrow \partial_n u^+ = 0$

$[\partial_n u] = [0'\varphi] = 0 \Rightarrow \partial_n u^- = 0 \Rightarrow u = \text{const}$

$\Delta u = 0$

$\varphi = [u] = [0\varphi]$

"$\supseteq$"

What conditions on the RHS do we get for int. Neumann and ext. Dirichlet?

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}\ldots$ do not know ψ. Assume $0 \in \Omega$.



$K(x,y) = \hat{n}_y \cdot \nabla G(x,y) \quad \rightsquigarrow \quad \tilde{K}(x,y) = \hat{n}_y \cdot \nabla G(x,y) + \frac{1}{\|x\|^{n-2}}$

$\hookrightarrow \tilde{D}\varphi(x) = \int \partial_{n_y} K(x,y)\,\varphi(y)\,dS_y + \frac{1}{\|x\|}\int \varphi(y)\,dS_y$

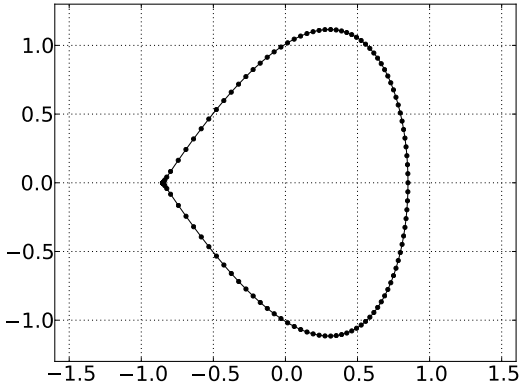
PDE, jump, compactness

$\frac{\varphi}{2} + \tilde{D}\varphi = 0 \Rightarrow \quad u(x) := \tilde{D}\varphi(x) \Rightarrow u^e = 0 \Rightarrow u \equiv 0$ on ext.

$0 = \|x\|^{n-2} u(x) = \int \varphi + O\left(\frac{1}{r}\right)$ as $r \to \infty \Rightarrow \int \varphi = 0$

$\varphi \equiv \text{const} \neq 0. \quad \Rightarrow \varphi \equiv 0$

Domains with Corners





What's the problem?

Domains with Corners (II)

At corner x_0: (2D)

$$\lim_{x \to x_0 \pm} = \int_{\partial\Omega} \hat{n} \cdot \nabla_y G(x,y)\phi(y) ds_y \pm \frac{1}{2} \frac{\langle \text{opening angle on } \pm \text{ side} \rangle}{\pi} \phi$$

Name some problems.

Workarounds?

Numerically: Needs consideration, can drive up cost through refinement.

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems
- Laplace
- Helmholtz
- Calderón identities

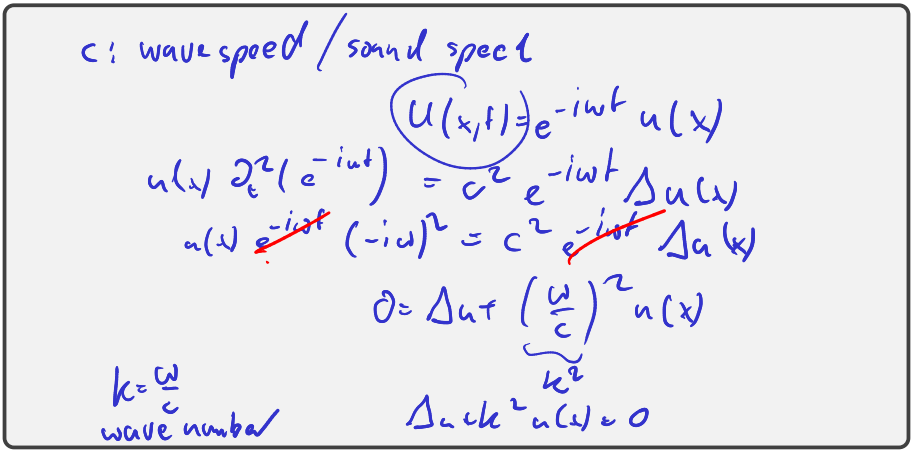
Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

Where does Helmholtz come from?

Derive the Helmholtz equation from the wave equation $\partial_t^2 U = c^2 \triangle U$,. Q: What is c?



Helmholtz vs. Yukawa

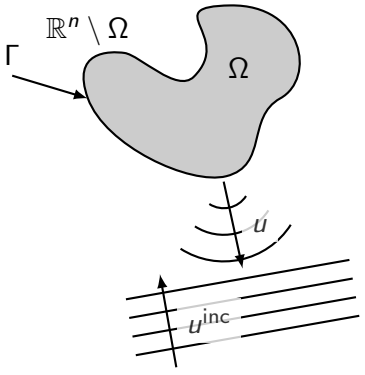
Helmholtz Equation

- $\triangle u + k^2 u(x) = 0$
- Indefinite operator
- Oscillatory solution
- Difficult to solve, especially for large k

Yukawa Equation

- $-\triangle u + k^2 u(x) = 0$
- Positive definite operator
- Smooth solutions
- 'Screened Coulomb' interaction
- Generally quite simple to solve

The prototypical Helmholtz BVP: A Scattering Problem



$\Delta u^{tot} + k^2 u^{tot} = 0$
$= inc =$
$= u =$

Ansatz:

$$u^{\text{tot}} = u + u^{\text{inc}}$$

Solve for scattered field u.

Helmholtz: Some Physics

Physical quantities:

- Velocity potential: $U(x,t) = u(x)e^{-i\omega t}$ (fix phase by e.g. taking real part)
- Velocity: $v = (1/\rho_0)\nabla U$
- Pressure: $p = -\partial_t U = i\omega u e^{-i\omega t}$
  - Equation of state: $p = f(\rho)$



What's ρ_0?

Density about which we linearize

What happens to a pressure BC as $\omega \to 0$?

Helmholtz: Boundary Conditions

Interfaces between media: What's continuous?

normal v, pressure

- Sound-soft: Scatterer "gives"
  - Pressure remains constant in time
  - $u = f \to$ Dirichlet
- Sound-hard: Scatterer "does not give"
  - Pressure varies, same on both sides of interface
  - $\hat{n} \cdot \nabla u = 0 \to$ Neumann
- Impedance: Some pressure translates into motion
  - Scatterer "resists"
  - $\hat{n} \cdot \nabla u + ik\lambda u = 0 \to$ Robin ($\lambda > 0$)
- Sommerfeld radiation condition: allow only outgoing waves (n-dim)

$$r^{\frac{n-1}{2}} \left(\frac{\partial}{\partial r} - ik \right) u(x) \to 0 \qquad (r \to \infty)$$

$\partial_t u + \partial_x u = 0$

Many interesting BCs $\to$ many IEs! :)

Unchanged from Laplace

Theorem (Green's Formula [Colton/Kress IAEST Thm 2.1])

If $\triangle u + k^2 u = 0$, *then*

$$(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} u(x) & x \in D \\ \frac{u(x)}{2} & x \in \partial D \\ 0 & x \notin D \end{cases}$$

$$[Su] = 0$$

$$\lim_{x \to x_0 \pm} (S'u) = \left(S' \mp \frac{1}{2} I\right)(u)(x_0) \quad \Rightarrow \quad [S'u] = -u$$

$$\lim_{x \to x_0 \pm} (Du) = \left(D \pm \frac{1}{2} I\right)(u)(x_0) \quad \Rightarrow \quad [Du] = u$$

$$[D'u] = 0$$

Unchanged from Laplace

Why is singular behavior (esp. jump conditions) unchanged?

Why does Green's formula survive?

Resonances



$-\triangle$ on a bounded (interior) domain with homogeneous Dirichlet/Neumann BCs has countably many real, positive eigenvalues.
What does that have to with Helmholtz?

Interior problems have resonances.

Why could it cause grief?