November 17, 2026

Announcements

- HW5 : Thu
- Project presentations

Goals

- IE solvability
- Calderón

- Numerics

Review

- IE solvability for Laplace: ✓
- Helmholtz.
  - have: • Green's formula
    • jump relations

$\Delta u + k^2 u = 0$

$\hookrightarrow G(\vec{x}, \vec{y}) = \begin{cases} H_0^{(1)}(kr) & 2D \\ \frac{e^{ikr}}{r} & 3D \end{cases}$

Helmholtz: Boundary Value Problems

$\Delta u + k^2 u = 0$
$u = 0$ on $\partial\Omega$

Find $u \in C(\bar{D})$ with $\Delta u + k^2 = 0$ such that

	Dirichlet	**Neumann**
Int.	$\lim_{x \to \partial D-} u(x) = g$ unique ($-$resonances)	$\lim_{x \to \partial D-} \hat{n} \cdot \nabla u(x) = g$ unique ($-$resonances)
Ext.	$\lim_{x \to \partial D+} u(x) = g$ Sommerfeld unique	$\lim_{x \to \partial D+} \hat{n} \cdot \nabla u(x) = g$ Sommerfeld unique

with $g \in C(\partial D)$.

Find layer potential representations for each.

Patching up spurious resonances inherited from adjoint

Issue: Exterior IE inherits non-uniqueness from 'adjoint' interior BVP.

$$u(x) := (D\varphi)(x) - i\alpha (S\varphi)(x) \qquad \text{"CFIE"}$$

$$g(x) = \lim_{x \to \partial\Omega_+} u(x) = \frac{\varphi(x)}{2} + (D\varphi)(x) - i\alpha (S\varphi)(x)$$

$\alpha = 1$. Suppose $\frac{\varphi}{2} + D\varphi - iS\varphi = 0$. $\lim_+ u = 0$ $u = 0$ on $\partial\Omega$

$\lim_+ \partial_n u = 0$

Patching up resonances: CFIE (1/3)

$$-\partial_n u^- - \partial_n u^+ - \partial_n u^- = [\partial_n u] = [\partial_n (D\varphi - iS\varphi)] = i\varphi$$

$$-u^- - u^+ - u^- = [u] = [D\varphi - iS\varphi] = \varphi$$

$$-i\,\partial_n u_- = u_-$$

Patching up resonances: CFIE (2/3)

$u\bar{u} = |u|^2$

$-i\partial_\nu u_- = u_-$

$\int_\Omega u \Delta v + \nabla u \cdot \nabla v = \int_{\partial\Omega} u\, \partial_n v$ with $v = \bar{u}$

$\int_\Omega u(-k^2\bar{u}) + |\nabla u|^2$

$= \underbrace{\int_\Omega -k^2 |u|^2 + |\nabla u|^2}_{\in \mathbb{R}} = \int u_- \partial_n \bar{u}_- = -i \int_{\partial\Omega} |u_-|^2$

$\int_{\partial\Omega} |u_-|^2 = 0$

$\Rightarrow u_- = 0,$

Patching up resonances: CFIE (3/3)

Via jump rel $0 = [u] = \varphi$

For ext. Neumann.

$$u(x) := S\sigma(x) + iD\sigma(x)$$

$g(x) = \lim_{x \to \partial\Omega_+} \partial_n u(x) = \left(-\frac{\sigma}{2} + S'\sigma + iD'\sigma \right)(x)$

Helmholtz Uniqueness

Uniqueness for remaining IEs similar:



Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems
- Laplace
- Helmholtz
- Calderón identities

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

D' is Self-Adjoint

Show that D' is self-adjoint. [Kress LIE 3rd ed. Sec 7.6]

$(D'\varphi, \psi) = (\varphi, D'\psi)$

$u(x) := D\varphi \qquad v(x) := D\psi$

Green's second thm., u, v satisfy Helmholtz

$$\int_{\partial\Omega} (\partial_n u) v_\pm = \int_{\partial\Omega} u_\pm \partial_n v_\pm$$

$$(\partial_n u_\pm, v_\pm) = (u_\pm, \partial_n v_\pm)$$

$$(D'\varphi, \psi) = (\partial_n u, [v]) = (\partial_n u, v_+) - (\partial_n u, v_-)$$

$$= (u_+, \partial_n v) - (u_-, \partial_n v)$$

$$= ([u], \partial_n v) = (\varphi, D'\psi)$$

Towards Calderón

Show that $(S\varphi, D'\psi) = ((S' + I/2)\varphi, (D - I/2)\psi)$. [Kress LIE 3rd ed. Sec 7.6]

$w := S\varphi \qquad v = D\psi$

$(S\varphi, D'\psi) = (w, \partial_n v) = (\partial_n w_-, v_-) =$

$(\varphi, SD'\psi)$?

$$
\begin{aligned}
(\varphi, SD'\psi) &= (S\varphi, D'\psi) \\
&= ((S' + \tfrac{1}{2})\varphi, (D - \tfrac{1}{2})\psi) \\
&= (\varphi, (D + \tfrac{1}{2})(D - \tfrac{1}{2})\psi) \\
&= (\varphi, (D^2 - \tfrac{1}{4})\psi)
\end{aligned}
$$

Calderón Identities: Summary

- $SD' = D^2 - I/4$
- $D'S = S'^2 - I/4$

Also valid for Laplace (jump relation same after all!)

Why do we care?

ext. Neumann:

$$u(x) := S\sigma(x) + iDS\sigma'(x)$$

$$g(x) = \lim_{x \to \partial\Omega} \partial_n u(x) = \left(-\tfrac{I}{2} + S'\sigma + iD'S\sigma\right)(x)$$

$(S')^2 - \frac{1}{4}$

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Back from Infinity: Discretization
- Fundamentals: Meshes, Functions, and Approximation
- Integral Equation Discretizations
- Integral Equation Discretizations: Nyström
- Integral Equation Discretizations: Projection

Computing Integrals: Approaches to Quadrature

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Numerics: What do we need?

- Discretize curves and surfaces
  - Interpolation
  - Grid management
  - Adaptivity
- Discretize densities
- Discretize integral equations
  - Nyström, Collocation, Galerkin
- Compute integrals on them
  - "Smooth" quadrature
  - Singular quadrature
- Solve linear systems

Constructing Discrete Function Spaces

Floating point numbers (*Degrees of Freedom* or *DoFs*) $\leftrightarrow$ Functions

Discretization relies on three things:

- Base/reference domain
- Basis of functions
- Meaning of DoFs

Related finite element concept: *Ciarlet triple*

Discretization options for a curve?

Closed curve: Fourier
or piecewise poly.

What do the DoFs mean?

Common DoF choices:

- Point values of function
- Point values of (directional?) derivatives
- Basis coefficients
- Moments

Often: useful to have both "modes", "nodes", jump back and forth

Why high order?

Order p: Error bounded as $|u_h - u| \leq Ch^p$

Thought experiment:



First order	Fifth order
1,000 DoFs $\approx$ 1,000 triangles	1,000 DoFs $\approx$ 66 triangles
Error: 0.1	Error: 0.1
Error: 0.01 $\rightarrow$?	Error: 0.01 $\rightarrow$?

Complete the table.

Remarks:

- Want $p \geq 3$ available.
- Assumption: Solution sufficiently smooth
- Ideally: p chosen by user