November 19, 2026

Announcements

- HW3 graded
- Projects

Goals

- discretization
  - show and tell : meshes
  - Nyström, projection

Review

- IEs for everybody!

Why high order?

Order p: Error bounded as $|u_h - u| \leq Ch^p$

Thought experiment:

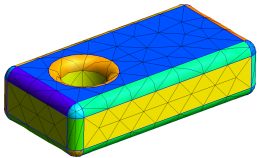
First order	Fifth order
1,000 DoFs $\approx$ 1,000 triangles	1,000 DoFs $\approx$ 66 triangles
Error: 0.1	Error: 0.1
Error: 0.01 $\rightarrow$?	Error: 0.01 $\rightarrow$?

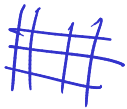
Complete the table.

Remarks:

- Want $p \geq 3$ available.
- Assumption: Solution sufficiently smooth
- Ideally: p chosen by user

What is an Unstructured Mesh?





Why have an unstructured mesh?

- complex topology
- corners
- adaptivity

What is the trade-off in going unstructured?

Demo: CAD software

Fixed-order vs Spectral

Fixed-order	Spectral
Number of DoFs n	Number of DoFs n
$\sim$	$\sim$
Number of 'elements'	Number of modes resolved
$\text{Error} \sim \frac{1}{n^p}$	$\text{Error} \sim \frac{1}{C^n}$
Examples?	Examples?
▶ Piecewise Polynomials	▶ Global Fourier
	▶ Global Orth. Polynomials

What assumptions are buried in each of these?

Fixed-order vs Spectral

What should the DoFs be?

What's the difficulty with purely modal discretizations?

Vandermonde Matrices

$$\begin{bmatrix} x_0^0 & x_0^1 & \cdots & x_0^n \\ x_1^0 & x_1^1 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n^0 & x_n^1 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = ?$$

Generalized Vandermonde Matrices

$$\begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \cdots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \cdots & \phi_n(x_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = ?$$

Generalized Vandermonde Matrices

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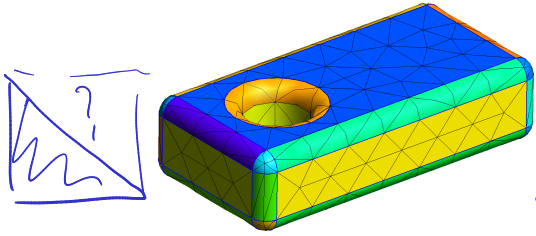
- Node placement? **Demo:** Interpolation node placement
- Vandermonde conditioning? **Demo:** Vandermonde conditioning
- What about multiple dimensions?
  - **Demo:** Visualizing the 2D PKDO Basis
  - **Demo:** 2D Interpolation Nodes

Common Operations

(Generalized) Vandermonde matrices simplify common operations:

- Modal $\leftrightarrow$ Nodal ("Global interpolation")
  - Filtering
  - Up-/Oversampling
- Point interpolation (Hint: solve using V^T)
- Differentiation
- Indefinite Integration
- Inner product
- Definite integration

Unstructured Mesh



- Design a data structure to represent this
- Compute normal vectors
- Compute area
- Compute integral of a function
- How is the function represented?

Demo: Working with Unstructured Meshes

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization
- Fundamentals: Meshes, Functions, and Approximation
- Integral Equation Discretizations
- Integral Equation Discretizations: Nyström
- Integral Equation Discretizations: Projection

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

Integral Equation Discretizations: Overview

$$\phi(x) - \int_\Gamma K(x,y)\phi(y)dy = f(y)$$

Nyström	Projection
▶ Approximate integral by quadrature: $\int_\Gamma f(y)dy \to \sum_{k=1}^n \omega_k f(y_k)$ ▶ Evaluate quadrature'd IE at quadrature nodes, solve	▶ Consider residual: $R := \phi - A\phi - f$ ▶ Pick projection P_n onto finite-dimensional subspace $P_n\phi := \sum_{k=1}^n \langle \phi, v_k \rangle w_k \quad \to$ DOFs $\langle \phi, v_k \rangle$ ▶ Solve $P_n R = 0$

Projection/Galerkin

- Equivalent to projection: Test IE with test functions
- Important in projection methods: *sub*-space (e.g. of $C(\Gamma)$)

Name some generic discrete projection bases.

polynomials,
Galerkin (test = trial)
collocation
Petrov-Galerkin ($\neq$)

Collocation and Nyström: the same?

Ny.: must be quadrature

Are projection methods implementable?

approximate integrals

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Nyström Discretizations (1/4)

Nyström consists of two distinct steps:

1. Approximate integral by quadrature:

$$\varphi_n(x) \approx \sum_{k=1}^{n} \omega_k K(x, y_k)\varphi_n(y_k) \neq f(x) \tag{1}$$

2. Evaluate quadrature'd IE at quadrature nodes, solve discrete system

$$\varphi_j^{(n)} - \sum_{k=1}^{n} \omega_k K(x_j, y_k)\varphi_k^{(n)} = f(x_j) \tag{2}$$

with $x_j = y_j$ and $\varphi_j^{(n)} = \varphi_n(x_j) = \varphi_n(y_j)$

Is version (1) solvable?

maybe not

Nyström Discretizations (2/4)

What's special about (2)?

Solution density also only known at point values. But: can get approximate continuous density. How?

Assuming the IE comes from a BVP. Do we also only get the BVP solution at discrete points?

Nyström Discretizations (3/4)

Does (1) $\Rightarrow$ (2) hold?

Does (2) $\Rightarrow$ (1) hold?

Nyström Discretizations (4/4)



What good does that do us?

Need to think in terms of functions!

Does Nyström work for first-kind IEs?

nope

Convergence for Nyström (1/2)

Increase number of quadrature points n:
Get sequence (A_n)
Want $A_n \to A$ in some sense
What senses of convergence are there for sequences of functions f_n?

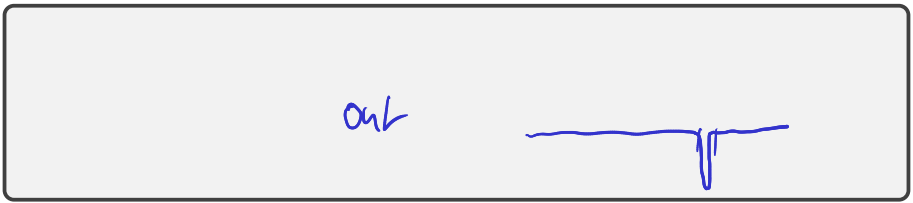
pointwise
uniform $\|f_n - f\|_\infty \to 0$

What senses of convergence are there for sequences of operators A_n?

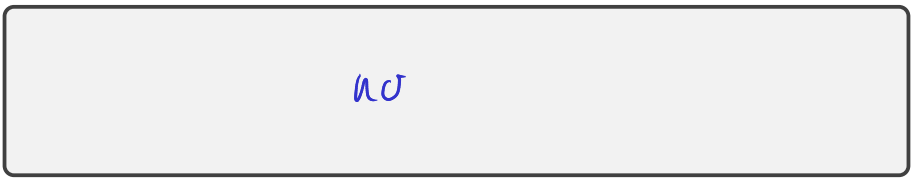
- "function wise"
- uniform $\|A - A_n\|$

Convergence for Nyström (2/2)

Will we get norm convergence $\|A_n - A\|_\infty \to 0$ for Nyström? [Kress LIE 3rd ed. Thm. 12.8]



Is functionwise convergence good enough?



Compactness-Based Convergence

X Banach space (think: of functions)

Theorem (Not-quite-norm convergence [Kress LIE 3rd ed. Cor. 10.7])

$A_n : X \to X$ bounded linear operators,
functionwise convergent to $A : X \to X$
Then convergence is uniform on compact subsets $U \subset X$, i.e.

$$\sup_{\phi \in U} \|A_n \phi - A\phi\| \to 0 \qquad (n \to \infty)$$

How is this different from norm convergence?

UBP

Collective Compactness

Set $\mathcal{A}$ of operators $A : X \to X$

Definition (Collectively compact)

$\mathcal{A}$ is called *collectively compact* if and only if
for $U \subset X$ bounded, $\mathcal{A}(U)$ is relatively compact.

What was relative compactness (=precompactness)?

Collective Compactness: Questions (1/2)

Is each operator in the set $\mathcal{A}$ compact?

Is collective compactness the same as "every operator in $\mathcal{A}$ is compact"?

Collective Compactness: Questions (2/2)

When is a sequence collectively compact?

Is the limit operator of such a sequence compact?

How can we use the two together?

Making use of Collective Compactness

X Banach space, $A_n : X \to X$, (A_n) collectively compact, $A_n \to A$ functionwise.

Corollary (Post-compact convergence [Kress LIE 3rd ed. Cor 10.11])

- $\|(A_n - A)A\| \to 0$
- $\|(A_n - A)A_n\| \to 0$
  $(n \to \infty)$

Anselone's Theorem

$(I - A)^{-1}$ exists, with $A : X \to X$ compact, $(A_n) : X \to X$ collectively compact and $A_n \to A$ functionwise.

Theorem (Nyström error estimate [Kress LIE 3rd ed. Thm 10.12])

For sufficiently large n, $(I - A_n)$ is invertible and

$$\|\phi_n - \phi\| \leq C(\|(A_n - A)\phi\| + \|f_n - f\|)$$

$$C = \frac{1 + \|(I - A)^{-1}A_n\|}{1 - \|(I - A)^{-1}(A_n - A)A_n\|}$$

$I + (I - A)^{-1}A = ?$

$\frac{\cdot 1}{1-q} = 1 + \frac{q}{1-q} = \frac{1-q+q}{1-q}$

$(I - A)^{-1}$

Anselone's Theorem: Proof (I)

Define approximate inverse $B_n = I + (I - A)^{-1} A_n$.

How good of an inverse is it?

$$
\begin{aligned}
\mathrm{Id} &\approx^{?} B_n(I - A_n) \\
&= (I + (I - A)^{-1} A_n)(I - A_n) \\
&= [I + (I - A)^{-1} A_n] - [A_n + (I - A)^{-1} A_n A_n] \\
&= [I + (I - A)^{-1} A_n] - [(I - A)^{-1}(I - A) A_n + (I - A)^{-1} A_n A_n] \\
&= [I + (I - A)^{-1} A_n] - [(I - A)^{-1} I A_n - (I - A)^{-1} A A_n + (I - A)^{-1} A_n A_n] \\
&= I + (I - A)^{-1} A A_n - (I - A)^{-1} A_n A_n \\
&= I + \underbrace{(I - A)^{-1}(A - A_n) A_n}_{-S_n} = I - S_n
\end{aligned}
$$

Anselone's Theorem: Proof (II)

Want $S_n \to 0$ somehow. Prior result gives us $\|(A - A_n)A_n\| \to 0$.

$\|S_n\| \to 0$

$\|S_n\| < 1$

$\|(I - S_n)^{-1}\| \leq \frac{1}{1 - \|S_n\|}$

$B_n(I - A_n) = I - S_n$

$(I - A_n)^{-1} = (I - S_n)^{-1} B_n$

Anselone's Theorem: Proof (III)

Let φ be the exact density that solves $(I - A)\varphi = f$ and φ_n the approximate density that solves $(I - A_n)\varphi_n = f_n$. Then consider

$$\begin{aligned}(I - A_n)(\varphi_n - \varphi) &= f_n - (I - A_n)\varphi \\ &= f_n - (I - A)\varphi + (A - A_n)\varphi \\ &= f_n - f + (A - A_n)\varphi\end{aligned}$$

Combining all this knowledge as

$$\begin{aligned}\|\varphi_n - \varphi\| &\leqslant \|(I - A_n)^{-1}\|(\|f_n - f\| + \|(A_n - A)\varphi\|) \\ &\leqslant \frac{\|B_n\|}{1 - \|S_n\|}(\|f_n - f\| + \|(A_n - A)\varphi\|)\end{aligned}$$

gives the desired estimate.