September 1, 2026

Announcements

- HW1 due tonight
- HW2

Goals

- Recap range finder
- approx theorem
- More factorizations
  - RRQR
  - Interpolative Decomposition

Review

- Low rank
  - how
  - how to use
  - range finder

QR too expensive

How do we construct the LRA basis?

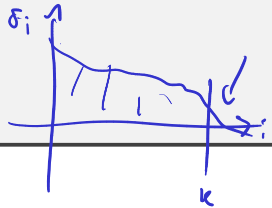
Put randomness to work: A $n \times n$ LR

Ω $n \times \ell$ Gaussian iid

$Y = A\Omega$

$Y = QR$

$A = U\Sigma V^T$



$Y_q = (AA^T)^q A\Omega$

Tweaking the Range Finder (I)

Can we accelerate convergence?

Tweaking the Range Finder (II)

What is one possible issue with the power method?

- overflow? orthogonality? QR after A, A^T

Even Faster Matvecs for Range Finding

Assumptions on Ω are pretty weak–can use more or less anything we want.
$\rightarrow$ Make it so that we can apply the matvec $A\Omega$ in $O(n^2 \log \ell)$ time.
How? Pick Ω as a carefully-chosen subsampling of the Fourier transform.
(many other approaches also exist)

Outline

Errors in Random Approximations

If we use the randomized range finder, how close do we get to the optimal answer?

Theorem (Halko/Tropp/Martinsson '10, (1.9))

For an $m \times n$ matrix A, a target rank $k \geqslant 2$ and an oversampling parameter $p \geqslant 2$ with $k + p \leqslant \min(m, n)$, with probability $1 - 6 \cdot p^{-p}$,

$$\left|A - QQ^T A\right|_2 \leqslant \left(1 + 11\sqrt{k+p}\sqrt{\min(m,n)}\right)\sigma_{k+1}.$$

(given a few more very mild assumptions on p)

What's the role of the oversampling parameter p?

- Better probabilistic guarantees!
- not accuracy.

HTM Proof Sketch: Rotate into singular-vector coordinates

Write the SVD and split after the first k singular directions.

$$A = U \begin{bmatrix} \overbrace{\Sigma_1}^{k} & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

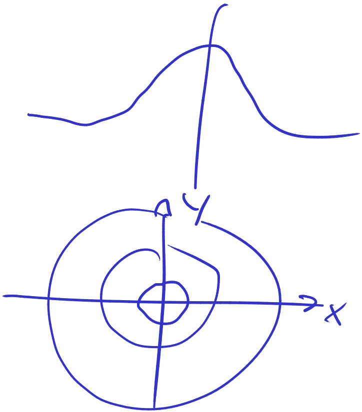
$$\Omega_1 = V_1^T \Omega \qquad \Omega_2 = V_2^T \Omega$$

$$Y = A\Omega = U \begin{bmatrix} \Sigma_1 \Omega_1 \\ \Sigma_2 \Omega_2 \end{bmatrix}$$

Ω_1, Ω_2 still Gaussian iid.

$$x \sim N(0,1)$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N(0,1)$$



HTM Proof Sketch: Isolating the Tail

$\|\Omega_2\|_2$

Let $P_Y = QQ^T$. Theorem 8.1 in [HTM '10] gives (X^+: pseudoinv.)

$$\|(I-P_Y)A\|_2^2 \le \|\Sigma_2\|^2 + \|\Sigma_2\Omega_2\Omega_1^{+}\|_2^2$$

σ_{k+1}^2 $\quad \sigma_{k+1}^2 \cdot (\quad \ldots \quad)$

Since $\|\Sigma_2\| = \sigma_{k+1}$, using $\sqrt{a^2+b^2} \le a+b$,

$$\|(I-P_Y)A\|_2 \le \|\Sigma_2\| + \|\Sigma_2\Omega_2\Omega_1^{+}\|_2$$

What do the factors capture?

$\Sigma_2 \rightsquigarrow \sigma_{k+1}$

$\Omega_2 \rightsquigarrow$ "tail sampling"

$\Omega_1^{+} \rightsquigarrow$ large if nearly missed some of V_1

HTM Proof Sketch: Gaussian pseudoinverse bounds

Ω_1 is a wide $k \times (k+p)$ Gaussian matrix. For $t \geq 1$, except with probability at most $5t^{-p}$,

$$\|\Omega_1^{+}\|_F \leq t\sqrt{\frac{12k}{p}}, \qquad \|\Omega_1^{+}\| \leq t\,\frac{e\sqrt{k+p}}{p+1}.$$

HTM Proof Sketch: Gaussian Concentration

Let G be a standard Gaussian matrix, and let f be L-Lipschitz with respect to the Frobenius norm:

$$|f(X) - f(Y)| \le L\|X - Y\|_F.$$

Then, for every $u \ge 0$,

$$\mathbb{P}\left[f(G) \geq \mathbb{E}[f(G)] + Lu \right] \leq e^{-u^2/2}$$

[HTM Prop. 10.3]

HTM Proof Sketch: Condition on the Leading Block

Fix Ω_1 and regard only Ω_2 as random. Define

$$h(X) = \|\Sigma_2 X \Omega_1^{+}\|.$$

A Gaussian product estimate yields

$$Eh(\Omega_2) \leq \|\Sigma_2\| \, \|\Omega_1^{+}\|_F + \|\Sigma_2\|_F \, \|\Omega_1^{+}\|.$$

Meanwhile,

$$|h(X) - h(Z)| \leq \underbrace{\|\Sigma_2\| \, \|\Omega_1^{+}\|}_{L} \, \|X - Z\|_F.$$

So h is Lipschitz, with an explicit constant L.

HTM Proof Sketch: A WIP Estimate

Gaussian concentration says $h(\Omega_2) \leq Eh(\Omega_2) + uL$ except with probability $e^{-u^2/2}$.

Combine with the pseudoinverse bounds:

$$\|(I - P_Y)A\| \leq \left(1 + t\sqrt{\frac{12k}{p}}\right)\sigma_{k+1} + t\frac{e\sqrt{k+p}}{p+1}\left(\sum_{j>k}\sigma_j^2\right)^{1/2} + ut\frac{e\sqrt{k+p}}{p+1}\sigma_{k+1}.$$

Failure probability: $5t^{-p} + e^{-u^2/2}$.

A-posteriori and Adaptivity

The HTM bound is *a-priori*. Once we're done, can we find out 'how well it turned out'?

$A - QQ^T A$

$E = (I - QQ^T)A$

$\|E\|_2 = \sigma_1(E)$

Grab random vector ω,

consider $\|E\omega\|_2 \sim \sigma_{\text{left}}$.