Fast Algorithms and Integral Equation Methods

CS598APK

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You've been lied to

$\mathcal{O}(n^2)$ is fine?

Conditioning

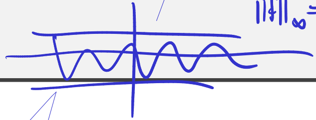
rel. error in out $\leq \kappa \cdot$ rel. error in input

$\kappa(A) = \|A\| \, \|A^{-1}\|$

$\partial_x : C^1 \to C^0$

$\|\cdot\|_\infty$

$\|\partial_x\| = \sup_{\|f\|_\infty = 1} \|\partial_x f\|$



$\rightsquigarrow$ Derivatives are unbounded.

What's the point of this class?

- Starting point: Large-scale scientific computing
- Many popular numerical algorithms: $O(n^\alpha)$ for $\alpha > 1$ (Think Matvec, Matmat, Gaussian Elimination, LU, ...)
- Build a set of tools that lets you cheat: Keep α small (Generally: probably not–Special purpose: possible!)
- Final goal: Extend this technology to yield PDE solvers
- But: Technology applies in many other situations
  - Many-body simulation
  - Stochastic Modeling
  - Image Processing
  - 'Data Science' (e.g. Graph Problems)
- This is class is about an even mix of math and computation

Integral Operators

$$A\vec{x} = \vec{b} \rightarrow K\vec{\sigma} = \vec{f}$$

$$\sum_j K_{ij}\sigma_j = f_i$$

$$\rightarrow \int_{\Omega'} K(x,y)\,\sigma(y)\,dy = f(x)$$

$i \leftrightarrow x$

$j \leftrightarrow y$

$$K(\vec{x},\vec{y}) = \frac{1}{\|\vec{x}-\vec{y}\|_2}$$

$$\sum K(\vec{x}, y_j)\,\sigma_j$$

Efficiency	Solvability	Applications
$O(n^2)$	$\rightarrow$ Theory	$\rightarrow$ PDE solvers

Solvability

$\sigma(y)$ $\qquad \sigma : \Omega' \to \mathbb{C}$

$$(A\sigma)(x) = \int_{\Omega'} k(x,y)\,\sigma(y)\,dy \qquad (x \in \Omega)$$

If k is smooth, A is a compact operator.

A smooth function is one that has a rapidly convergent Taylor expansion

well-approx. by an operator with finite-dim. range

Solvability

$A\sigma = f$ ← no bueno

compact

integral equation of the first kind

$(I + A)\sigma = f$ ← second-kind equation

Fredholm alternative

Efficiency

- Low-rank approximation
    - smooth functions → short expansion
      → low rank

$$A = \vec{u}\vec{v}^T \qquad A\vec{x} = \vec{u}\left(\vec{v}^T\vec{x}\right)$$

- $\int k(x,y)\,\sigma(y)\,dy$

$\int k(x-y)\,\sigma(y)\,dy$ ← convolution

Efficiency

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\} \cdot \mathcal{F}(g)$$

convolution structure

Applications: (Elliptic) PDE solving (I)

Applications: (Elliptic) PDE solving (II)

And one more thing...

Both multigrid and fast/IE schemes ultimately are $O(N)$ in the number of degrees of freedom N.

Survey

- Home dept
- Degree pursued
- Longest program ever written
  - in Python?
- Research area
- Interest in PDE solvers

Class web page

https://tinyurl.com/fastalg-f26

contains:

- Class outline
- Notes
- Demos
- Assignments
- Discussion forum
- Grading
- Video

Open Source <3

These notes (and the accompanying demos) are open-source!

Bug reports and pull requests welcome:
https://github.com/inducer/fast-alg-ie-notes

Sources

- Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions by Halko/Martinsson/Tropp
- Carrier, Greengard, Rokhlin: A Fast Adaptive Multipole Algorithm for Particle Simulations
- Rainer Kress: Linear integral equations. (second edition)
- David Colton and Rainer Kress: Inverse Acoustic and Electromagnetic Scattering Theory. (3rd edition)