August 27, 2026

Announcements

Goals

- PDE solvers from those pieces
- logistics
- operators in action

Review

- integral operators
- "fast" algorithms
- compact operators

Applications: (Elliptic) PDE solving (I)

$\partial_x^2 + \partial_y^2$

Poisson: $\Delta u = f$ (on Ω) $\quad u = g$ (on $\partial\Omega$)

$\Delta G = \delta \in$ fundamental solution (+ BCs at ∞)

$\Delta G(x) = 0 \quad (x \neq 0)$

$\int \delta \, 1 = 1$

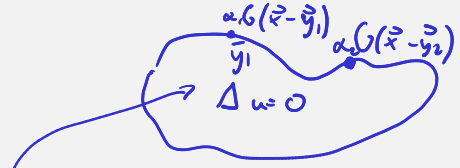
$\int \Delta G \, \varphi \, dx = \int \delta \varphi \, dx = \varphi(0) \qquad \varphi \in C^\infty$

amenable to integration by parts

Applications: (Elliptic) PDE solving (I)

$G(\vec{x}) = \frac{1}{4\pi} \frac{1}{\|x\|_2}$ $\quad \Delta u = \rho$

Laplace: $\Delta u = 0$ (on Ω) $\quad u = g$ (on $\partial\Omega$)



$u(\vec{x}) = \alpha_1 G(\vec{x}-\vec{y}_1) + \alpha_2 G(x-\vec{y}_2) + \cdots$

$= \sum_i \alpha_i G(\vec{x}-\vec{y}_i)$ point potential

layer potential $\rightsquigarrow u(\vec{x}) = \int G(\vec{x}-\vec{y})\,\alpha(y)\,d\vec{y}$

Applications: (Elliptic) PDE solving (I)

$$S\sigma(x) = \int G(x-y)\alpha(y)\,dS_y = g(x) \qquad (x \in \partial\Omega)$$

$$\tilde{u}(x) = \int_\Omega G(\vec{x}-\vec{y})\, f(\vec{y})\, d\vec{x}$$

volume potential

$$= G * f$$

$$\Delta(G * f) = f$$

$$\Delta\tilde{u}(x) = (\Delta G(x - \cdot)) * f$$

$$\Delta\tilde{u} = f$$

$$= (\delta(x - \cdot)) * f$$

$$= f$$

$$u_{poisson} = \tilde{u} + \hat{u}$$

$$\Delta\hat{u} = 0$$

$$g(x) = u_{poisson}(x) = \tilde{u} + \hat{u}$$

Applications: (Elliptic) PDE solving (II)

And one more thing...

Both multigrid and fast/IE schemes ultimately are $O(N)$ in the number of degrees of freedom N.

Survey

- Home dept
- Degree pursued
- Longest program ever written
  - in Python?
- Research area
- Interest in PDE solvers

Class web page

https://tinyurl.com/fastalg-f26

contains:

- Class outline
- Notes
- Demos
- Assignments
- Discussion forum
- Grading
- Video

Open Source <3

These notes (and the accompanying demos) are open-source!

Bug reports and pull requests welcome:
https://github.com/inducer/fast-alg-ie-notes

Sources

- Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions by Halko/Martinsson/Tropp
- Carrier, Greengard, Rokhlin: A Fast Adaptive Multipole Algorithm for Particle Simulations
- Rainer Kress: Linear integral equations. (second edition)
- David Colton and Rainer Kress: Inverse Acoustic and Electromagnetic Scattering Theory. (3rd edition)

Outline

Matvec: A Slow Algorithm

Matrix-vector multiplication: our first 'slow' algorithm.
$O(N^2)$ complexity.

$$\beta_i = \sum_{j=1}^{N} A_{ij}\alpha_j$$

Assume A dense.

Matrices and Point Interactions

$$A_{ij} = G(x_i, y_j)$$

row col row col.

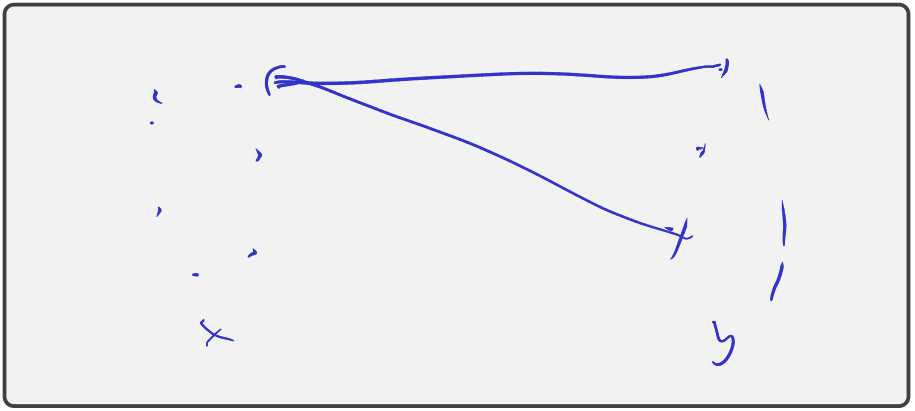
Does that actually change anything?

x_i: target
y_j: source
G: kernel

Matrices and Point Interactions

$$A_{ij} = G(x_i, y_j)$$

Graphically, too:



Matrices and point Interactions

$$\psi(x_i) = \sum_{j=1}^{N} G(x_i, y_j)\varphi(y_j)$$

This *feels* different.

Q: Are there enough matrices that come from globally defined G to make this worth studying?

Point Interaction Matrices: Examples (I)

- (Lagrange) interpolation

$$\gamma(x) = \sum_{j=1}^{M} \ell_j(x)\, \varphi(y_j)$$

  Interpolation error

- Numerical diff.

$$\gamma(x) = \sum_{j=1}^{M} \ell_j'(x)\, \varphi(y_j)$$

- Quadrature $\gamma(x) \approx$ . . .

Point Interaction Matrices: Examples (II)

- Potential kernels
- Fourier
- Special function transforms (e.g. spherical harmonic)
- Convolutions

Point Interaction Matrices: Examples (III)

So yes, there are indeed lots of these things.

Integral Operators

Why did we go through the trouble of rephrasing matvecs as

$$\psi(x_i) = \sum_{j=1}^{N} G(x_i, y_j)\varphi(y_j)?$$

$$\psi(x) \to \int G(x,y)\varphi(y)\,dy$$

Cheaper Matvecs

$$\psi(x_i) = \sum_{j=1}^{N} G(x_i, y_j)\varphi(y_j)$$

So what can we do to make evaluating this cheaper?

Fast Dense Matvecs

Consider

$$A_{ij} = u_i v_j,$$

let $\boldsymbol{u} = (u_i)$ and $\boldsymbol{v} = (v_j)$.
Can we compute $A\boldsymbol{x}$ quickly? (for a vector $\boldsymbol{x}$)

$A = \vec{u}\vec{v}^T$

$A\vec{x} = (\vec{u}\vec{v}^T)\vec{x} = \vec{u}(\vec{v}^T\vec{x})$

$A\varphi = u(x)\int v(y)\varphi(y)\,dy$