

September 1, 2026

Announcements

- HW1

Goals

- Low rank approximation
 - Use (SVD)
 - Forming
 - theoretical
 - practical

Review

Low-Rank Point Interaction Matrices

Usable with low-rank complexity reduction?

$$\psi(x_i) = \sum_{j=1}^N G(x_i, y_j) \varphi(y_j)$$

$$\vec{y} = A \vec{x}$$

$$\Psi(x) =$$

$$\vec{y} = \begin{bmatrix} \vec{u}_1 & \vec{u}_1^T \end{bmatrix} \vec{x}$$

$$\Psi(x) = \left\{ \int_i u(x) v(y) \varphi(y) dy \right.$$

separation of
variables

$$\Psi(x) = \int G(x, y) \varphi(y) dy$$

Numerical Rank

What would a *numerical* generalization of 'rank' look like?

RRET

$$A = U \cdot \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right] \cdot V^T$$

Eckart-Young-Mirsky Theorem

$$\xi = \begin{pmatrix} \lambda_1 & & \\ & \lambda_k & \\ & & 0 \end{pmatrix} \geq 0$$

Theorem (Eckart-Young-Mirsky)

SVD $A = U\Sigma V^T$. If $k < r = \text{rank}(A)$ and

$$A_k = \sum_{i=1}^k \sigma_i u_i v_i^T,$$

then

$$\min_{\text{rank}(B)=k} |A - B|_2 = |A - A_k|_2 = \sigma_{k+1}.$$

Q: What's that error in the Frobenius norm?

So in principle that's good news:

- ▶ We can find the numerical rank.
- ▶ We can also find a factorization that reveals that rank (!)

Demo: Rank of a Potential Evaluation Matrix (Attempt 2)

Constructing a tool

There is still a slight downside, though.

- Build matrix $O(n^2)$
- SVD $O(n^3)$

Outline

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Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

- Low-Rank Approximation: Basics

- Low-Rank Approximation: Error Control

- Reducing Complexity

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

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Rephrasing Low-Rank Approximations

SVD answers low-rank-approximation ('LRA') question. But: too expensive. First, rephrase the LRA problem:

$$\underset{n \times h}{A} \approx \underset{n \times k}{B} \underset{k \times n}{C^T} \quad \leftarrow \text{factorization form}$$

$$\underset{n \times h}{A} \approx \underset{\substack{\uparrow \\ n \times k}}{Q} Q^T A \quad \leftarrow \text{projection form}$$

orthonormal columns

Using LRA bases

$$\begin{matrix} A & : & n \times n \\ Q & , & n \times k \end{matrix}$$

If we have an LRA basis Q , can we compute an SVD?

$$\textcircled{1} \quad C = Q^T A$$

$$\textcircled{2} \quad \text{SVD } C = \bar{U} \Sigma V^T$$

$$\textcircled{3} \quad U = Q \bar{U}$$

$$A \approx Q Q^T A = \underbrace{Q \bar{U}}_U \Sigma V^T$$

$$m \times n$$

$$m \times n$$

$$D$$

$$m^2 n \quad mn^2 \leftarrow (\text{Golub-Reinsch?})$$

cost

$$n^2 k$$

$$n k^2$$

$$n k^2$$

Finding an LRA basis

How would we *find* an LRA basis?



Giving up optimality

What problem should we actually solve then?

$$\min_{\text{rank}(B)=k} \|A - B\|_2 = \|A - A_k\|_2 \approx \sigma_{k+1}.$$

Recap: The Power Method

How did the power method work again?

$$A \vec{x} = \lambda \vec{x}$$

$\vec{x}_1, \dots, \vec{x}_n$
 $\lambda_1, \dots, \lambda_n$
 $|\lambda_1| \geq |\lambda_2| \geq \dots$

$$A \vec{x}$$

$$\vec{x} = \alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2 + \dots + \alpha_n \vec{x}_n$$

$$A \vec{x} = \alpha_1 \lambda_1 \vec{x}_1 + \alpha_2 \lambda_2 \vec{x}_2 + \dots + \alpha_n \lambda_n \vec{x}_n$$

How do we construct the LRA basis?

Put randomness to work:

- ① Draw an $n \times \ell$ Gaussian iid matrix Ω
- ② $Y = A\Omega$
- ③ $Y = Q \underset{\uparrow}{R}$

Tweaking the Range Finder (I)

Can we accelerate convergence?

~~$$Y = A^9 \Sigma$$~~

$$Y = (A A^T)^9 A \Sigma$$

$$A = U \Sigma V^T$$

$$\begin{aligned} & U \Sigma V^T V \Sigma V^T U \Sigma V^T \\ &= U \Sigma^3 V^T \end{aligned}$$

Tweaking the Range Finder (II)

What is one possible issue with the power method?



Even Faster Matvecs for Range Finding

Assumptions on Ω are pretty weak—can use more or less anything we want.
→ Make it so that we can apply the matvec $A\Omega$ in $O(n^2 \log \ell)$ time.
How? Pick Ω as a carefully-chosen subsampling of the Fourier transform.
(many other approaches also exist)

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Errors in Random Approximations

If we use the randomized range finder, how close do we get to the optimal answer?

Theorem (Halko/Tropp/Martinsson '10, (1.9))

For an $m \times n$ matrix A , a target rank $k \geq 2$ and an oversampling parameter $p \geq 2$ with $k + p \leq \min(m, n)$, with probability $1 - 6 \cdot p^{-p}$,

$$\left| A - QQ^T A \right|_2 \leq \left(1 + 11\sqrt{k+p}\sqrt{\min(m,n)} \right) \sigma_{k+1}.$$

(given a few more very mild assumptions on p)

What's the role of the oversampling parameter p ?